THE

ELEMENTS of ALGEBRA,

IN TEN BOOKS:

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THE ELEMENTS of ALGEBRA

BOOK VI.

Of such problems as usually pass under the name of Diophantine problems.

Of DIOPHANTUS *and his writings.*

223. DIOPHANTUS of *Alexandria* in *Egypt* is the first writer of Algebra we meet with among the Ancients: not that the invention of that art is particularly to be ascribed to him; for besides that we meet with some few sketches of it in authors of greater antiquity, *Diophantus* has no where, that I know of, taught the fundamental principles and rules of Algebra: he treats it every where as an art already known, and seems to intend, not so much to teach, as to cultivate and improve it, by applying it to certain indeterminate arithmetical problems concerning square and cube numbers, right-angled triangles, *&c*, which till that time seemed to have been either not at all considered, or at least not regularly treated of. These problems are very curious and entertaining; but yet in the resolution of them there frequently occur difficulties, which nothing less than the nicest and most refined Algebra, applied with the utmost skill and judgment could ever surmount: and most certain it is, that in this way, no man ever extended the limits of the analytic art further than *Diophantus* has done, or discovered greater penetration and judgment; whether we consider his wonderful sagacity and peculiar artifice in forming such proper positions as the nature of the questions under consideration required, or the more than ordinary subtilty of his reasoning upon them. Every particular problem puts us upon a new way of thinking, and furnishes a fresh vein of analytical treasure, which, considering the vast variety there is of them, cannot but be very instructive to the mind in conducting itself through almost all difficulties of this kind, wherever they occur. Of these problems I shall se-

lect ſome, to wit, ſo many as may ſuffice to give the learner a juſt taſte of this ſort of reaſoning, and to incite him to look into the author himſelf, whenever he ſhall find leiſure enough for that purpoſe.

The way of ſubſtituting letters for quantities given in a problem, as well as for thoſe that are ſought, was either not known, or not in uſe in *Diophantus*'s time; nor the way of introducing more letters than one into a problem where more unknown quantities were to be repreſented: and (to confeſs the truth) the intolerable liberty ſome modern Analyſts have taken in this way, makes *Diophantus*'s method appear much the more elegant of the two; though it muſt not be denied but that it ſometimes occaſions perplexity and confuſion, when the ſame letter in one and the ſame problem is made to ſignify two or three different things, as it is employed by *Diophantus* in ſo many different operations. For my own part, I ſhall ſteer a middle courſe (as often as I ſee occaſion) between theſe two extremes, and ſhall endeavour, as far as I can, to dreſs up theſe problems in ſuch a manner as may render them agreeable to a modern taſte, without loſing any of the elegancies of *Diophantus*'s *analyſis*. Upon the whole, if it be allowed that theſe problems have not ſuffered in paſſing through my hands, that will ſufficiently juſtify my inſerting them here, as others before me in their algebraical tracts have done; and that is the utmoſt I ſhall pretend to.

As to the age *Diophantus* lived in, we have nothing certain: all that his learned and beſt commentator Monſieur *Bachet* could collect concerning this point* is as follows.

The famous Geometrician *Apollonius* flouriſhed in the time of *Ptolemæus Euergetes* king of *Egypt*, and conſequently above two hundred years before Chriſt. *Hypſicles*, who is ſuppoſed to be the author of the fourteenth and fifteenth books annexed to *Euclid*'s elements, tells us in his preface to thoſe two books, that he had ſeen two copies of a treatiſe of *Apollonius* concerning the regular bodies, whereof the former was very imperfect, but the latter came from *Apollonius* himſelf: whence it plainly appears that *Hypſicles* muſt have lived not long after *Apollonius*. *Diophantus* in his treatiſe of multangular numbers quotes *Hypſicles*; and therefore muſt have lived ſome time after him. On the other hand, the learned lady *Hypatia* who was daughter to *Theo* the Mathematician, and lived in the time of *Arcadius* and *Honorius*, that is, about four hundred years after Chriſt, wrote a comment upon *Diophantus*; and therefore *Diophantus* muſt have lived a conſiderable time before her, ſince it is not uſual to write comments upon modern productions. All then that can reaſonably be conjectured from this account is, that *Diophantus* flouriſhed in or about the third century.

His

His Arithmetics, out of which treatise these problems are collected, consisted of thirteen books, whereof only the first six books are now extant. The best edition I have met with of *Diophantus*'s works is that published at *Paris* by Monsieur *Bachet* in the year 1621: he has certainly best explained *Diophantus*'s meaning, and made the best observations upon him, of any commentator before him; besides that, he has restored and cleared up the text in innumerable places, which by the ignorance or negligence of transcribers, was in almost every problem most miserably corrupted. Besides that abovementioned, there is another edition of *Bachet*'s *Diophantus*, published in the year 1670, with *Fermat*'s observations upon some few of the questions.

LEMMA.

224. *If all the three sides of a right-angled triangle be increased or diminished in the same proportion, there will be formed another right-angled triangle similar to the former.*

That the triangle thus found will be similar to the former, needs no proof; since the similitude of triangles consists in nothing else but in a proportionality of their sides, that is, when every side of one triangle hath the same proportion to it's correspondent side in another: and from hence it follows, that the triangle formed as above described will be a right-angled triangle, since all similar triangles are equiangular. But without touching upon geometrical principles, I shall demonstrate this in the following manner.

Let a, b and c be the three sides of a right-angled triangle, to wit, a and b the legs, or sides about the right-angle, and c the hypotenuse; then will $a^2+b^2=c^2$. Let now all the three sides a, b and c be increased or diminished in the proportion of d to e, by saying, as d is to e so is a to $\frac{ae}{d}$; also, as d is to e so is b to $\frac{be}{d}$; lastly, as d is to e so is c to $\frac{ce}{d}$; this done, I say that the numbers $\frac{ae}{d}$, $\frac{be}{d}$ and $\frac{ce}{d}$ will be the three sides of a right-angled triangle: for the square of $\frac{ae}{d}$ is $\frac{a^2e^2}{d^2}$, and the square of $\frac{be}{d}$ is $\frac{b^2e^2}{d^2}$, and the sum of these two squares is $\frac{a^2e^2+b^2e^2}{d^2}=\frac{c^2e^2}{d^2}$, because $a^2+b^2=c^2$; therefore the numbers $\frac{ae}{d}$ and $\frac{be}{d}$ will represent the two legs of a right-angled triangle whose hypotenuse is $\frac{ce}{d}$. *Q. E. D.*

COROLLARY.

Hence if a right-angled triangle be given, another similar to it may be formed whose hypotenuse shall be any assigned number whatever. Ex. gr. Let a, b and c represent the sides of a given right-angled triangle whose hypotenuse is c, and let it be required to form another similar to this whose hypotenuse shall be f: this may be done by saying, *As* c, *the hypotenuse of the original triangle, is to* f, *the hypotenuse of the triangle to be formed, so is* a, *one of the legs of the former triangle, to* $\frac{af}{c}$ *a correspondent leg of the latter triangle; and so is* b *the other leg of the former triangle to* $\frac{bf}{c}$ *the other leg of the latter triangle:* The three sides therefore of the latter triangle will be $\frac{af}{c}$, $\frac{bf}{c}$ and f, or $\frac{af}{c}$, $\frac{bf}{c}$ and $\frac{cf}{c}$.

PROBLEM I.

Being the 8th of the second book of Diophantus's *Arithmetics.*

225. *It is required to divide a given square number into two such parts, that each part may be a square number.*

N. B. By numbers, in a simple sense, are meant by *Diophantus* all rational numbers, whether they be whole numbers or fractions, in contradistinction to surd roots, and all such other numbers as are incommensurable to unity, which are commonly called irrational numbers.

SOLUTION.

Let 100 be the square number proposed to be divided; and since the two parts of this number are also to be squares, let xx be one of them, and the other will be $100 - xx$, which, as well as the former, is to be a square. Now as the problem supposes no relation betwixt the sides of these two squares, we are left at liberty to feign what relation we please betwixt them, provided that will but furnish us with a proper equation for determining the side of the first square. Let us then suppose the side of the second square to be $2x - 10$, (the reason of which position will be explained hereafter;) then must the square of $2x - 10$ be equal to the second square, that is, to $100 - xx$: but the square of $2x - 10$ is $4xx - 40x + 100$; therefore $4xx - 40x + 100 = 100 - xx$; resolve this equation by striking off 100 from both sides &c, and you will have x, the root of the first square, equal to 8; whence $2x - 10$,

or

or the ſide of the ſecond ſquare, will be 6; therefore 8 and 6 will be the ſides of the two ſquares ſought, and the ſquares themſelves will be 64 and 36, which both together make 100, as required.

The Explication.

1ſt. If any one aſks why $2x-10$ is here ſubſtituted for the ſecond ſide rather than y or any other unknown quantity; it may be anſwered, that the problem furniſhes only one equation, to wit, $x^2+y^2=100$; and therefore it will be impoſſible to determine either x or y without feigning ſome other equation expreſſing the relation betwixt x and y, as by making $y=2x-10$, *&c.*

2dly. If it be aſked why the ſide of the ſecond ſquare was feigned $2x-10$ rather than $2x-11$, $2x-9$, or any other; the anſwer is, becauſe 10 is the ſide or root of 100, the ſquare firſt propoſed to be divided; and therefore by this poſition we have 100 on both ſides of the equation; which being ſtruck off from both ſides leaves a ſimple equation, wherein x will always be rational: whereas had any other number been joined with $2x$, the equation would have been a quadratic, and would not (except by chance) have exhibited x rational. Thus if $2x-11$ had been made the ſide of the ſecond ſquare, the equation would have been $4xx-44x+121=100-xx$.

3dly. If it be aſked why the ſide of the ſecond ſquare was made $2x-10$ rather than $10-2x$; I anſwer, that either of theſe two poſitions would equally have ſerved the purpoſe, becauſe the ſquare of $10-2x$ is the ſame with the ſquare of $2x-10$, and therefore the equation would have been the ſame in both caſes. It muſt not be omitted however, that if $2x-10$ be an affirmative quantity, $10-2x$ will be a contrary negative, and *vice verſa*: but this can create no difficulty; for if $10-2x$ had been found negative, it might eaſily have been exchanged for the affirmative ſide, and the ſquare ſought would ſtill have been the ſame.

4thly. If it be aſked why $2x-10$ rather than $2x+10$; I anſwer, that this laſt poſition might have been uſed; but then the equation would have been $4xx+40x+100=100-xx$; in which caſe x (the ſide of the firſt ſquare) would have been found equal to -8, and $2x+10$ (the ſide of the ſecond) equal to -6: however theſe negative ſides -8 and -6 might eaſily have been exchanged for their affirmatives $+8$ and $+6$, as was before obſerved. It appears however from what has been ſaid, that $2x-10$ was a more elegant ſubſtitution.

5thly. If it be aſked why $2x-10$ was made the ſide of the ſecond ſquare rather than $x-10$, $3x-10$, $4x-10$, $5x-10$, *&c*; the anſwer is, that had $x-10$ been made the ſide of the ſecond ſquare, the equation

equation would have been $xx-20x+100=100-xx$, in which case x, the side of the first square, would have been found equal to 10, and $x-10$, or the side of the second, equal to nothing: had the side of the second square been put equal to $3x-10$, the equation would have given us the same squares as before, but in an inverted order, that is, x would now have been found equal to 6, and $3x-10=8$: had $4x-10$, or $5x-10$ been used to express the side of the second square, these positions would have been less simple, and the squares produced by them fractional numbers, which nevertheless would have answered the conditions of the question, and in some of *Diophantus*'s questions are unavoidable.

To form a general canon for the resolution of any question of this kind, *Let* aa *be the square to be divided*; then assuming at pleasure any two numbers, r a greater and s a less, let sx be the side of one of the squares sought, and let $rx-a$ be the side of the other; then will the former square be s^2x^2, and the latter $r^2x^2-2arx+a^2=a^2-s^2x^2$; resolve this last equation, and you will have $x=\frac{2ar}{r^2+s^2}$: hence $rx-a$, the side of one of the squares sought, will be $\frac{2ar^2}{r^2+s^2}-\frac{a}{1}=\frac{ar^2-as^2}{r^2+s^2}$; and sx, the side of the other square, will be $\frac{2ars}{r^2+s^2}$: therefore *If any two unequal numbers be taken at pleasure, whereof the greater is called* r *and the lesser* s, *the sides of the two squares sought will be* $\frac{ar^2-as^2}{r^2+s^2}$ and $\frac{2ars}{r^2+s^2}$. The reason why r and s must be unequal is, because if it was otherwise, the side $\frac{ar^2-as^2}{r^2+s^2}$ would be nothing, as it would be negative if r was not made to signify the greater of the two numbers r and s.

This problem admits of an infinite number of solutions, according to the different significations of r and s.

EXAMPLE I.

Let $r=2$ and $s=1$; then we shall have $\frac{ar^2-as^2}{r^2+s^2}=\frac{3a}{5}$, and $\frac{2ars}{r^2+s^2}=\frac{4a}{5}$. Now the square of $\frac{3a}{5}$ is $\frac{9aa}{25}$, and the square of $\frac{4a}{5}$ is $\frac{16aa}{25}$, and the sum of these two squares is $\frac{25aa}{25}$ or aa, as required.

SCHOLIUM.

SCHOLIUM.

Since $\frac{ar^2 - as^2}{r^2 + s^2}$ and $\frac{2ars}{r^2 + s^2}$ are the sides of two squares, which squares when added together make a^2, it follows, that if $\frac{ar^2 - as^2}{r^2 + s^2}$ and $\frac{2ars}{r^2 + s^2}$ be made the two legs of a right-angled triangle, the hypotenuse of that triangle will be a, and so we shall have a right-angled triangle whose three sides are all expressed in rational numbers. Let now these three sides be all increased or diminished in the proportion of a to $r^2 + s^2$, that is, let all the sides be multiplied by $r^2 + s^2$ and divided by a, and the resulting numbers will be $r^2 - s^2$, $2rs$, and $r^2 + s^2$; but according to the foregoing lemma, these three numbers will also represent the three sides of another right-angled triangle whose hypotenuse is $r^2 + s^2$: hence then we are fallen into the same method for forming a right-angled triangle out of any two given numbers whatever, as is laid down in article 12, which see: the rule is this. *Assuming any two unequal numbers whatever* r *and* s, *whereof* r *is the greater, make* $r^2 - s^2 = d$, $2rs = e$, *and* $r^2 + s^2 = f$: *then will* d, e *and* f *be the three sides of a right-angled triangle formed from the numbers* r *and* s, *whose hypotenuse is* f. According to this way of notation, the sides of the two squares that solved the foregoing problem, to wit, $\frac{ar^2 - as^2}{r^2 + s^2}$ and $\frac{2ars}{r^2 + s^2}$ will now be $\frac{ad}{f}$ and $\frac{ae}{f}$; whence arises another canon for the resolution of the foregoing problem, which is this.

Let d, e *and* f *be the three sides of any right-angled triangle whose hypotenuse is* f; *and by coroll. in art.* 224 *find another triangle similar to this, whose hypotenuse is* a; *then will the two legs of this last triangle be the sides of two squares which will answer the condition of the problem,* because these two sides will be $\frac{ad}{f}$ and $\frac{ae}{f}$, as above.

PROBLEM 2.

Being the 10*th of the second book of* Diophantus.

226. *It is required to divide any number consisting of two square numbers into other two square numbers.*

N. B. This problem is of great use in *Diophantus*, being made the *basis* or foundation of the resolution of several other problems; and therefore I must be excused if I insist somewhat the longer upon it. I shall here give it a general solution, that the reasons of the positions and the limitations of the problem may the more easily appear; or where they do not, may the more easily be explained.

SOLUTION.

Let then $\overline{a^2+b^2}$ be the number to be divided, being composed of two square numbers, to wit, a^2 a greater, and b^2 a less; and let it be required to divide this number into other two square numbers. To do this, take two unequal known numbers, r a greater and s a less, but with these two cautions notwithstanding, to wit, that the proportion of r to s be not the same with that of a to b, nor the same with that of $a+b$ to $a-b$. These precautions being observed, make $rx-a$ the side of one of the squares sought, and $sx-b$ the side of the other; then will the former square be $r^2x^2-2arx+a^2$, and the latter $s^2x^2-2bsx+b^2$, and the sum of these two squares is $r^2x^2+s^2x^2-2arx-2bsx+a^2+b^2=a^2+b^2$ by the supposition; resolve this equation, and you will have $x=\frac{2ar+2bs}{r^2+s^2}$: hence $rx-a$, the side of the first square, will be $\frac{2ar^2+2brs}{r^2+s^2}-\frac{a}{1}=\frac{ar^2-as^2+2brs}{r^2+s^2}$. From the two numbers r and s form (by art. 12) a right-angled triangle whose two legs and hypotenuse let be d, e and f respectively, by making $r^2-s^2=d$, $2rs=e$, and $r^2+s^2=f$; then will $\frac{ar^2-as^2+2brs}{r^2+s^2}$ (or the side of the first square) be $\frac{ad+be}{f}$. Again, since $x=\frac{2ar+2bs}{r^2+s^2}$, we shall have $sx-b$, the side of the second square, equal to $\frac{2ars+bss-brr}{r^2+s^2}$, that is, according to our former notation, $\frac{ae-bd}{f}$. Thus then we have the sides of two squares which will solve the problem, to wit, $\frac{ad+be}{f}$ and $\frac{ae-bd}{f}$: and this theorem arises by making $rx-a$ and $sx-b$ the sides of the two squares sought.

But suppose the side of the second square had been made $sx+b$ instead of $sx-b$, what then would have been the consequence? why, without repeating the foregoing operation, upon this supposition it is easy to foresee, that all the difference in the conclusion will be this, to wit, that those terms in the foregoing theorem wherein b was concerned must now have the signs changed, and then you will have a theorem calculated for this latter supposition: but the former theorem was, that $\frac{ad+be}{f}$ and $\frac{ae-bd}{f}$ were the sides of two squares which would solve the problem; therefore the

the theorem will be, that $\frac{ad-be}{f}$ and $\frac{ae+bd}{f}$ will be the ſides of another pair of ſquares which would equally ſolve the problem: thus we ſee the problem is capable of two ſolutions without changing the triangle d, e, f. But here it muſt be obſerved, that if any of the ſides of the ſquares expreſſed above come out negative, they muſt be changed into their affirmatives; which may be done, for reaſons given in the foregoing problem.

Had $rx+a$ and $sx+b$ been made to expreſs the ſides of the ſquare ſought, they would have come out the ſame as in the firſt caſe; eſpecially after the negative ſides in both caſes were made affirmative. Had $rx+a$ and $sx-b$ expreſſed the two ſides ſought, thoſe ſides would then have come out the ſame as in the ſecond caſe: ſo that though there be four caſes, there are but two theorems exhibiting two different ſolutions of the problem, which two theorems are both included in the following canon.

Let a *and* b *be the ſides of the original ſquares, whereof the number to be divided is ſuppoſed to conſiſt; and from any two unequal numbers, whereof the greater is not to the leſs as the greater of the two ſides* a *and* b *is to the leſs, nor as their ſum to their difference, let a right-angled triangle be formed, whereof the two legs and hypotenuſe are the numbers* d, e *and* f *reſpectively: multiply now the two legs* d *and* e *by* a *the ſide of one of the original ſquares, and put down the two products* ad *and* ae: *multiply again the ſame legs* d *and* e *in the ſame order, by* b *the ſide of the other original ſquare, and the two products* bd *and* be *put down after the two former; ſo that the products with the common denominator* f *placed under them may ſtand thus,* $\frac{ad}{f}, \frac{ae}{f}, \frac{bd}{f}, \frac{be}{f}$: *I ſay then, that of theſe four fractions the ſum of the extremes and the difference of the two middle terms will be the ſides of two ſquares which will ſolve the problem: I ſay moreover, that the difference of the extremes and the ſum of the two middle terms will be the ſides of other two ſquares which will equally ſolve the problem.*

N. B. The legs of the triangle may be taken in any order, to wit, d and e, or e and d in the firſt multiplication, provided they be taken in the ſame order in the ſecond multiplication: neither is it of any conſequence which of the two multiplicators a and b you begin with: the ſides of the ſquares ſought will come out the ſame in all theſe caſes, but in a different order, as will eaſily appear upon tryal.

EXAMPLE.

The number 13 is compoſed of two ſquare numbers, *viz*, 4 and 9, which I call original ſquares, whoſe ſides are 2 and 3: let it then be re-

 quired

quired to divide this number into two other ſquare numbers. Firſt then I aſſume two numbers, ſuppoſe 2 and 1, which are not in proportion to one another as 3 to 2, or as 5 to 1; and from theſe numbers 2 and 1, I form a right-angled triangle, whoſe ſides therefore will be 3, 4 and 5; then I multiply the legs of this triangle, to wit 3 and 4, by 2 the ſide of one of the original ſquares, and the products are 6 and 8; the ſame legs I multiply again by 3, the ſide of the other original ſquare, and the products are 9 and 12; theſe products being put down after the former with the common denominator 5 under them, I have four fractions, to wit, $\frac{6}{5}$, $\frac{8}{5}$, $\frac{9}{5}$ and $\frac{12}{5}$; the ſum of the extremes whereof is $\frac{18}{5}$, and the difference of the middle terms $\frac{1}{5}$; therefore $\frac{18}{5}$ and $\frac{1}{5}$ are the ſides of two ſquares which will ſolve the problem, as appears upon tryal; for the ſquare of $\frac{18}{5}$ is $\frac{324}{25}$, and the ſquare of $\frac{1}{5}$ is $\frac{1}{25}$, and their ſum is $\frac{325}{25}$ or 13. Again, the difference of the extreme fractions is $\frac{6}{5}$, and the ſum of the two middle terms is $\frac{17}{5}$; therefore $\frac{6}{5}$ and $\frac{17}{5}$ are the ſides of other two ſquares which will alſo ſolve the problem; for the ſquare of $\frac{6}{5}$ is $\frac{36}{25}$, and the ſquare of $\frac{17}{5}$ is $\frac{289}{25}$, and their ſum is $\frac{325}{25}$ or 13: here then we have two ſolutions from one and the ſame triangle, to wit, whoſe ſides are 3, 4 and 5; and any other triangle that is not ſimilar to this, and that is formed according to the directions above given, will furniſh two more ſolutions of the ſame problem; and ſo on *ad infinitum*.

If the number to be divided be compoſed of two equal ſquares, the two ſolutions which the ſame right-angled triangle otherwiſe affords will then run into one. As for inſtance, the number 2 is compoſed of two equal ſquares, to wit, 1 and 1, whoſe ſides are 1 and 1: let it then be required to divide this number 2 into other two ſquares. Firſt then, obſerving the precautions above given, I take the two numbers 2 and 1, whereby I form the right-angled triangle whoſe ſides are 3, 4 and 5, as above; then multiplying the legs 3 and 4 firſt by 1, and then again by 1, (which two multiplicators are the equal ſides of the original ſquares,) the products, with the common denominator 5 under them, will be $\frac{3}{5}$, $\frac{4}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, the ſum of the extremes whereof is $\frac{7}{5}$, and the difference of the

the middle terms $\frac{1}{5}$; therefore $\frac{7}{5}$ and $\frac{1}{5}$ will ſolve the problem, the ſquare of $\frac{7}{5}$ being $\frac{49}{25}$, and the ſquare of $\frac{1}{5}$, $\frac{1}{25}$, whoſe ſum is $\frac{50}{25}$ or 2: on the other hand, if the difference of the extremes and the ſum of the middle terms be taken, the ſides will be $\frac{1}{5}$ and $\frac{7}{5}$ the ſame as before, but in a contrary order.

The reaſon of the precautions above given in forming the right-angled triangle which is to ſolve the problem is this: If the two numbers forming the triangle be to each other as the ſides of the original ſquares, or as the ſum and difference of thoſe ſides, the triangle will give us two pair of ſquares, as before; but then one of theſe pairs will be the ſame with thoſe very original ſquares whereof the number to be divided is compoſed. As for example; as the number 13 conſiſts of two ſquares 4 and 9, whoſe ſides are 2 and 3, let it again be required to divide the number 13 into other two ſquares. To do this, let us take two numbers which are to each other as 2 to 3 the ſides of the original ſquares; nay let us take the very numbers 2 and 3, and by them form a right-angled triangle; and the ſides of the triangle will be 5, 12 and 13; multiply the legs 5 and 12 by the ſides 2 and 3, and the four products, with the common denominator 13, will be $\frac{10}{13}$, $\frac{24}{13}$, $\frac{15}{13}$ and $\frac{36}{13}$, whereof the ſum of the extremes is $\frac{46}{13}$, and the difference of the middle terms is $\frac{9}{13}$; and theſe ſides $\frac{46}{13}$ and $\frac{9}{13}$ will ſolve the problem, as may eaſily be ſeen: on the other hand, the difference of the extremes is $\frac{26}{13}$ or 2, and the ſum of the middle terms $\frac{39}{13}$ or 3; and the ſides 2 and 3 will ſolve the problem, but then they are the ſides of the original ſquares.

To demonſtrate this in general, let a and b be the ſides of the two original ſquares, to wit, a the greater and b the leſs; alſo let r and s be the two numbers from which the right-angled triangle is to be formed: and firſt let r be to s as a is to b; then by multiplying extremes and means we have $br=as$; multiply both ſides by $2s$, and you will have $2brs=2ass$; add $arr-ass$ to both ſides of the equation, and you will have $arr-ass+2brs=arr+ass$, that is, according to our former

former notation, $ad+be=af$; divide by f, and you will have $\frac{ad+be}{f}$, that is, one of the ſides ſought, equal to a, the ſide of one of the original ſquares; therefore the other ſide ſought muſt be b, the ſide of the other original ſquare; otherwiſe the ſum of the ſquares of theſe two ſides would not compoſe the number to be divided.

Again, let r be to s as $a+b$ to $a-b$; and by multiplying extremes and means we ſhall have $ar-br=as+bs$, and by tranſpoſition, $ar=as+br+bs$, and again by tranſpoſition, $ar-as=br+bs$; multiply both ſides by $r+s$, and you will have $ar^2-as^2=br^2+2brs+bs^2$; tranſpoſe $2brs$, and you will have $ar^2-as^2-2brs=br^2+bs^2$, that is, according to our former notation, $ad-be=bf$; divide by f, and you will have $\frac{ad-be}{f}$, which is one of the ſides ſought, equal to b, the ſide of one of the original ſquares.

SCHOLIUM.

The deſign of this ſecond problem was to divide any number as $\overline{a^2+b^2}$, conſiſting of two ſquare numbers, into other two ſquare numbers. Let the ſide b, and conſequently one of the original ſquares b^2 be equal to nothing; then will the problem be changed into this: *To divide a given ſquare number as* a^2 *into two other ſquares*, which is the firſt problem; therefore the firſt problem is but a particular caſe of the ſecond, to wit, when b in the ſecond problem equals 0; therefore if in the expreſſions of the ſides ſought in the ſecond problem, b be made equal to nothing, that is, if all the terms be ſtruck out wherein b is a multiplier, the reſt will expreſs the ſides ſought in the firſt problem; but $\frac{ad+be}{f}$ and $\frac{ae-bd}{f}$ expreſs the two ſides ſought in the ſecond problem; therefore $\frac{ad}{f}$ and $\frac{ae}{f}$ will expreſs the two ſides ſought in the firſt problem; and and ſo you will find them expreſſed, if you look back upon the reſolution of that firſt problem. The ſame expreſſions might alſo have been obtained from the other ſolution of this problem, where the ſides were $\frac{ad-be}{f}$ and $\frac{ae+bd}{f}$.

PROBLEM 3.

227. *To find four right-angled triangles expreſſed in whole numbers, which have all the ſame hypotenuſe.*

This

This problem is not in *Diophantus*, but is in a manner ſuppoſed by him in the twentyſecond queſtion of the third book of his Arithmetics.

Note, that in the reſolution of this problem we ſhall denote all right-angled triangles by their ſides, the hypotenuſe being always laſt expreſſed. Thus a, b, c denotes a right-angled triangle whoſe two legs are a and b, and whoſe hypotenuſe is c.

SOLUTION.

1ſt. Form two right-angled triangles that are not ſimilar to each other, ſuppoſe a, b, c, and d, e, f: theſe I call primitive triangles, as being thoſe from which the four triangles ſought are to be derived.

2dly. Multiply a, b, c, the three ſides of the firſt primitive triangle by f, the hypotenuſe of the other, and the products af, bf, cf will conſtitute another right-angled triangle whoſe hypotenuſe is cf, by art. 224; and this will be one of the four triangles ſought.

3dly. Multiply d, e, f, the three ſides of the ſecond primitive triangle, by c, the hypotenuſe of the firſt, and the products cd, ce, cf will conſtitute another right-angled triangle whoſe hypotenuſe is cf, and which therefore will be a ſecond triangle of the four ſought.

4thly. Multiply d and e, the legs of the ſecond primitive triangle, ſucceſſively by a and b, thoſe of the firſt, and the products with the common denominator f under them, being put down as in the canon to the ſecond problem, will ſtand thus, $\frac{ad}{f}, \frac{ae}{f}, \frac{bd}{f}, \frac{be}{f}$; the ſum of the extremes is $\frac{ad+be}{f}$, and the difference of the middle terms $\frac{ae-bd}{f}$ or $\frac{bd-ae}{f}$; therefore according to the canon above cited, the ſquare of $\frac{ad+be}{f}$ and the ſquare of $\frac{ae-bd}{f}$ muſt both together make a^2+b^2 or c^2; therefore $\frac{ad+be}{f}$ and $\frac{ae-bd}{f}$ muſt be the two legs of a right-angled triangle whoſe hypotenuſe is c; therefore, by art. 224, $\overline{ad+be}$ and $\overline{ae-bd}$ will be the legs of a right-angled triangle whoſe hypotenuſe is cf; therefore $\overline{ad+be}$, $\overline{ae-bd}$, cf will be the third triangle ſought.

5thly. The difference of the extremes of the abovewritten fractions is $\frac{ad-be}{f}$ or $\frac{be-ad}{f}$, and the ſum of the middle terms is $\frac{ae+bd}{f}$; therefore in the ſame manner as in the laſt paragraph it may be demonſtrated that $\overline{ad-be}$, $\overline{ae+bd}$, cf will be the fourth triangle ſought.

Thus

Thus then from the two primitive triangles firſt given we have derived four others, all expreſſed in whole numbers, and all having the ſame hypotenuſe, to wit,

1ſt, af, bf, cf.
2dly, cd, ce, cf.
3dly, $\overline{ad+be}$, $\overline{ae-bd}$, cf.
4thly, $\overline{ad-be}$, $\overline{ae+bd}$, cf.

EXAMPLE.

Let the primitive triangles be 3, 4, 5, formed by the numbers 2 and 1, and 5, 12, 13, formed by the numbers 3 and 2. Here if we multiply 3, 4 and 5, the three ſides of one of the original triangles, by 13 the hypotenuſe of the other, we ſhall have 39, 52, 65 for the firſt of the four triangles: again, if we multiply 5, 12 and 13 the three ſides of the other primitive triangle by 5 the hypotenuſe of the firſt, we ſhall have 25, 60, 65 for the ſecond triangle ſought: let us now multiply 5 and 12, the legs of the ſecond primitive triangle, by 3 and 4, the legs of the firſt, ſucceſſively, and the products will be 15, 36, 20, 48, whereof the ſum of the extremes is 63, and the difference of the middle terms 16; therefore 16, 63, 65 will be the third triangle: laſtly, the difference of the extremes is 33, and the ſum of the middle terms 56; therefore 33, 56, 65 will be our fourth triangle: ſo that from the two primitive right-angled triangles 3, 4, 5 and 5, 12, 13, we have derived four others expreſſed in whole numbers and having the ſame hypotenuſe, to wit,

1ſt, 39, 52, 65.
2dly, 25, 60, 65.
3dly, 16, 63, 65.
4thly, 33, 56, 65.

It was cautioned in forming the two primitive triangles, that they ſhould not be ſimilar to each other; becauſe if they be ſo, the firſt and ſecond of the triangles ſought will become one and the ſame triangle, and of the two laſt one will vaniſh. To demonſtrate this, let the triangle a, b, c be ſimilar to the triangle d, e, f, and let a and d, b and e, c and f be correſpondent ſides; then by art. 224, a will have the ſame proportion to d that c hath to f; whence we ſhall have $cd=af$: moreover b will be to e as c is to f; whence ce will be equal to bf; therefore cd, ce, cf, the ſides of the ſecond triangle ſought, will be the ſame with af, bf, cf, thoſe of the firſt: moreover from the ſuppoſed ſimilitude of the two primitive triangles a, b, c and d, e, f, we ſhall have a to d as b to e, and conſequently $ae=bd$, and $ae-bd=0$: but $ae-bd$ is one of the legs of

of the third triangle; therefore in this case, the third triangle will vanish; and we shall only have two right-angled triangles having the same hypotenuse, to wit, the first and the last.

LEMMA.

228. *In every right-angled triangle, if the double product of the legs be either added to or subtracted from the square of the hypotenuse, both the sum and remainder will be square numbers.*

Let a, b and c be the three sides of any right-angled triangle whose hypotenuse is c; I say then, that both c^2+2ab and c^2-2ab will be square numbers: for $c^2=a^2+b^2$; therefore $c^2+2ab=a^2+2ab+b^2$ a square number, whose side or root is $a+b$: in like manner $c^2-2ab=a^2-2ab+b^2$ a square number, whose side is $a-b$. *Q. E. D.*

PROBLEM 4.

Being one particular case of the twentysecond of the third book of Diophantus.

229. *To find a number with this property, to wit, that whether it be added to, or subtracted from it's square, both the sum and the remainder shall be square numbers.*

First SOLUTION.

Let x be a number that hath the property above described, that is, let x be such a number, that both $xx+x$ and $xx-x$ be squares, which we usually express thus, $xx+x=\square$, and $xx-x=\square$. This is one kind of that which *Diophantus* calls duplicate equality, to wit, when two different quantities are both to be equated to squares. Now to resolve this duplicate equality we are here to take notice, that the two quantities $xx+x$ and $xx-x$ both together make $2xx$; therefore if we can find two square numbers which both together make $2xx$, and if we make $xx-x$ equal to the lesser of these two squares, the other quantity $xx+x$ must necessarily be equal to the greater square; and so we shall have both $xx+x$ and $xx-x$ square numbers: now the number 2 was divided into two square numbers in the second problem, and those squares were $\frac{1}{25}$ and $\frac{49}{25}$; therefore $\frac{1}{25}+\frac{49}{25}=2$; therefore $\frac{xx}{25}+\frac{49xx}{25}=2xx$; therefore we have two square numbers which both together make $2xx$: let us then make $xx-x=\frac{xx}{25}$ the lesser of those

two

two squares, or $xx+x=\frac{49xx}{25}$ the greater; and either of these equations being resolved will give $x=\frac{25}{24}$; therefore the fraction $\frac{25}{24}$ will answer the conditions of the problem, which we may examine thus: the square of $\frac{25}{24}$ is $\frac{625}{576}$, and the side of that square $\frac{25}{24}$, by multiplying both the numerator and denominator by 24, becomes $\frac{600}{576}$; now if the side $\frac{600}{576}$ be added to it's square $\frac{625}{576}$, the sum $\frac{1225}{576}$ will be a square number, whose side is $\frac{35}{24}$: and on the other hand, if the side $\frac{600}{576}$ be subtracted from it's square $\frac{625}{576}$, there will remain $\frac{25}{576}$ a square number, whose side is $\frac{5}{24}$.

To the solution already given of this problem I shall add another, which for it's elegancy deserves a place here, and much more as it will prepare the learner for the better understanding of the next following problem.

Second SOLUTION.

Let a be the double product of the base and perpendicular of any right-angled triangle whose hypotenuse is b; then will both b^2+a and b^2-a be square numbers by the last article; and if these numbers be multiplied by any other square number, suppose xx, not as yet determined, the products $b^2x^2+ax^2$ and $b^2x^2-ax^2$ will still be square numbers, since a square multiplying a square produces a square. If therefore we wanted a number, which being added to, and subtracted from some square number would make both the sum and remainder squares, ax^2 would be the number, and b^2x^2 would be the square; and that, whatever the quantity x was made to signify: but we want a number which being added to, and subtracted from the square of itself will make both the sum and remainder square numbers; therefore to answer the requisites of this problem, the value of x must be such, that b^2x^2 may be the square of ax^2; but b^2x^2 is the square of bx; therefore ax^2 must be equal to bx: be it so, and we shall have $x=\frac{b}{a}$, and $xx=\frac{b^2}{a^2}$, and ax^2 (or the number sought) equal to $\frac{b^2}{a}$: *If therefore the square of the hypotenuse*

of

of any right-angled triangle be divided by the double product of the legs, the quotient will be such a number as is described in the problem. As for instance, 3, 4 and 5 are the sides of a right-angled triangle, the square of whose hypotenuse is 25, and the double product of whose base and perpendicular is 24; therefore $\frac{25}{24}$ is a number which will answer the conditions of the problem, as above.

PROBLEM 5.

Being the same with the twentysecond of the third book of Diophantus.

230. *To find four such numbers as being severally added to and subtracted from the square of their sum, will make both the sums and the remainders all square numbers.*

SOLUTION.

By the directions given in the third problem (art. 227) compute four right-angled triangles having all the same hypotenuse, which common hypotenuse call h; then taking these triangles in any order, let a, b, c, d represent the double products of their bases and perpendiculars multiplied together; that is, let a be the double product of the base and perpendicular of the first triangle multiplied together, and so of the rest; then by art. 228, $h^2 \pm a$, $h^2 \pm b$, $h^2 \pm c$, and $h^2 \pm d$ will all be square numbers; and other square numbers of the same stamp may be found out *ad infinitum*, by multiplying these into other square numbers taken at pleasure: let then x^2 be a square number hereafter to be determined, and let the square numbers already mentioned be all multiplied into x^2, and we shall have $h^2x^2 \pm ax^2$, $h^2x^2 \pm bx^2$, $h^2x^2 \pm cx^2$, and $h^2x^2 \pm dx^2$ all square numbers: therefore if we only wanted four such numbers, as being severally added to and subtracted from some square number, would make both the sums and remainders all squares, the numbers ax^2, bx^2, cx^2 and dx^2 would answer this purpose, and h^2x^2 would be the square number to and from which these numbers were to be added and subtracted; this would be the case, let the indeterminate quantity x be what it will; but we want four such numbers as being severally added to and subtracted from the square of their own sum, will make both the sums and remainders all square numbers; therefore to answer this condition, the quantity x must now be restrained to some particular signification, to wit, such a one as will make h^2x^2 the square of the sum of all the numbers above mentioned, that is, h^2x^2 must be the square of $\overline{ax^2+bx^2+cx^2+dx^2}$;

$\overline{cx^2+dx^2}$; or if we make $\overline{a+b+c+d}=e$, b^2x^4 must be the square of ex^2: but b^2x^2 is already the square of bx; therefore ex^2 must be equal to bx; whence we shall have $x=\frac{b}{e}$, and $x^2=\frac{b^2}{e^2}$, and ax^2, bx^2, cx^2 and dx^2, or the four numbers sought, equal to $\frac{ab^2}{e^2}$, $\frac{bb^2}{e^2}$, $\frac{cb^2}{e^2}$ and $\frac{db^2}{e^2}$ respectively. Q. E. D.

To demonstrate this synthetically, the sum of all these four numbers is $\frac{ab^2+bb^2+cb^2+db^2}{e^2}=\frac{eb^2}{e^2}=\frac{b^2}{e}$; therefore the square of their sum is $\frac{b^4}{e^2}$; add to this square number the first number $\frac{ab^2}{e^2}$, and the sum will be $\frac{b^4+ab^2}{e^2}=\frac{b^2}{e^2}\times\overline{b^2+a}$; but $\frac{b^2}{e^2}$ is in it's own nature a square number, and $\overline{b^2+a}$ is a square number by art. 228; therefore $\frac{b^2}{e^2}\times\overline{b^2+a}$ is a square number; that is, the first of these four numbers last found being added to the square of their sum will make a square number: and after the same manner it may be demonstrated that if the first number be subtracted from the square of their sum, the remainder will be a square number; and the same demonstration may also be applied to all the rest of the numbers; and therefore $\frac{ab^2}{e^2}$, $\frac{bb^2}{e^2}$, $\frac{cb^2}{e^2}$ and $\frac{db^2}{e^2}$ are such numbers as will answer the conditions of the problem.

If any one will be at the trouble of exemplifying this canon in numbers, he may make use of the four triangles already computed in the solution of the third problem, whose common hypotenuse is 65.

PROBLEM 6.

Being the 11th of the second book of Diophantus.

231. *To find two square numbers whose difference shall be any number given.*

This problem may be solved various ways; but the most elegant as well as the most useful solution is that which follows.

SOLUTION.

Let it be required to find two square numbers whose difference is d; and let this difference d be resolved into any two unequal factors a and b,

b, that is, let a and b be any two unequal numbers which being multiplied together will produce d, and whereof a is ſuppoſed the greater and b the leſs: put x for the ſide of the leſſer ſquare ſought; and ſince the difference of the ſides is not determined in the problem like the difference of the ſquares themſelves, call the ſide of the greater ſquare $x+b$; then will the ſquares themſelves be xx and $xx+2bx+bb$; and the difference of theſe two ſquares will be $2bx+bb=d=ab$; but if $2bx+bb=ab$, then dividing by b, we ſhall have $2x+b=a$, and x the ſide of the leſſer ſquare equal to $\frac{a-b}{2}$; whence $x+b$, the ſide of the greater ſquare will be $\frac{a+b}{2}$; and ſo the ſides of the two ſquares ſought will be $\frac{a-b}{2}$ and $\frac{a+b}{2}$. Hence may be derived the following canon:

Reſolve the given difference into any two unequal factors; and half the difference and half the ſum of thoſe factors will be the ſides of two ſquares whoſe difference is the number given.

As for example, let it be required to find two ſquare numbers whoſe difference is 60: now the number 60 may be reſolved into theſe two factors, 1 and 60, becauſe $1\times60=60$; the half difference of theſe two factors is $\frac{60-1}{2}$ or $\frac{59}{2}$, and half their ſum is $\frac{60+1}{2}$ or $\frac{61}{2}$; therefore $\frac{59}{2}$ and $\frac{61}{2}$ are the ſides of two ſquares whoſe difference is 60: and ſo we find them; for the ſquare of $\frac{59}{2}$ is $\frac{3481}{4}$, and the ſquare of $\frac{61}{2}$ is $\frac{3721}{4}$, and their difference is $\frac{240}{4}=60$. The number 60 may alſo be reſolved into other factors, as 2 and $\frac{60}{2}$, that is, 2 and 30; 3 and $\frac{60}{3}$, that is, 3 and 20; 4 and $\frac{60}{4}$, that is, 4 and 15; 5 and $\frac{60}{5}$, that is, 5 and 12; 6 and $\frac{60}{6}$, that is, 6 and 10; 7 and $\frac{60}{7}$, and ſo on *ad infinitum*; and therefore problems of this kind admit of an infinite number of ſolutions.

Again, let it be required to find two ſquare numbers whoſe difference is 45: now the number 45, amongſt an infinite number of other factors, may be reſolved into theſe; 1 and 45, 3 and 15, 5 and 9: the firſt couple, 1 and 45 give 22 and 23 for the ſides; the ſecond couple, 3 and

15 give 6 and 9 for the ſides; and the third couple, 5 and 9 give 2 and 7 for the ſides of two ſquares whoſe difference is 45.

Note, that if the two factors into which the given difference is reſolved be whole numbers, and both even or both odd, the ſides of the two ſquares ſought will come out in whole numbers, otherwiſe not; as may eaſily be ſeen by the two foregoing examples.

COROLLARY.

Hence, if there be two indeterminate quantities x *and* y, *whoſe difference* x — y *is reſolved into two factors* a *and* b, *ſuch, that the ſquare of* $\frac{a+b}{2}$ *may equal* x; *then will the ſquare of* $\frac{a-b}{2}$ *equal* y: *and* e converſo, *if the ſquare of* $\frac{a-b}{2}$ *equals* y, *then the ſquare of* $\frac{a+b}{2}$ *will equal* x; *and ſo in both caſes* x *and* y *will be ſquare numbers:* inſtances whereof will be ſeen in ſome of the following problems.

If the quantities x and y be ſo undetermined, that it cannot be known which is the greater, in ſuch a caſe either quantity may be ſuppoſed the greater, and the other may be ſubtracted from it; but then care muſt be taken that the ſquare of half the ſum of the factors be always equated to the ſuppoſed greater quantity, and the ſquare of half their difference to the leſs: and as to the factors themſelves, it matters not which is taken for the greater; for half their ſum will ſtill be the ſame, and the ſquare of half their difference will alſo be the ſame, whether that difference be expreſſed affirmatively or negatively.

LEMMA.

232. *If* a *and* b *be any two quantities whereof* aa *is greater than* b; *I ſay then, that* a *muſt either be greater than* $+\sqrt{b}$ *or leſs than* $-\sqrt{b}$.

As if b be 25 and a^2 be greater than 25, a muſt either be greater than $+5$ or leſs than -5.

For if a be made equal to any number within thoſe limits, as to ± 4, aa will be equal to 16, and conſequently will be leſs than 25, contrary to the ſuppoſition: whereas on the other hand, if a be made equal to any number without thoſe limits $+5$ and -5, as if a be made equal to ± 6, aa will be 36, and ſo will be greater than 25, agreeable to the ſuppoſition. In like manner it may be demonſtrated that *If* aa *be leſs than* b, a *muſt be ſome number between* $+\sqrt{b}$ *and* $-\sqrt{b}$.

PROBLEM 7.

Being the 12th of the ſecond book of Diophantus.

233. *To find a number, which being ſeverally added to two given numbers, will make them both ſquares.*

SOLU-

SOLUTION.

Let the given numbers be aa *and* bb, *whereof let* aa *be the greater, and let their difference be* d; not that it is neceſſary that the two given numbers ſhould be ſquares, ſince this condition is neither expreſſed nor implied in the problem; but they may however in a general calculation, to avoid ſurds, be repreſented under that form, as will eaſily be ſeen in the application annexed.

Put x for the number ſought, and then adding it ſeverally to the two given numbers aa and bb, we muſt have $x+aa$ and $x+bb$ both ſquare numbers. Here then we have another ſort of duplicate equality different from that in art. 229, and which therefore muſt be reſolved after a different manner: now to reſolve this duplicate equality, ſince the difference betwixt $x+aa$ and $x+bb$ is d, it follows, that if by art. 231 we find two ſquare numbers whoſe difference is d, and make $x+aa$ equal to the greater of theſe two ſquares, $x+bb$ muſt of courſe be equal to the leſs; or if $x+bb$ be made equal to the leſs, $x+aa$ muſt be equal to the greater, and ſo we ſhall have $x+aa$ and $x+bb$ both ſquare numbers. But it is not any two ſquare numbers whoſe difference is d that will ſerve our turn neither; for ſince the greater ſquare muſt be equal to $x+aa$, and the leſs to $x+bb$, it follows, that the greater ſquare muſt be greater than a^2, and that the leſs ſquare muſt be greater than b^2; but if the leſs ſquare be made greater than b^2, the other muſt of courſe be greater than a^2, becauſe the difference of the ſquares is the ſame with the difference of the numbers; therefore we are not only to find two ſquare numbers whoſe difference is d, but alſo ſuch, that the leſſer of thoſe two ſquares may be greater than bb: now two ſquares whoſe difference is d may be found by article 231, that is, by reſolving the difference d into two unequal factors, and by making half the difference and half the ſum of thoſe factors the ſides of the two ſquares ſought; and if any two ſquares would have ſerved our turn, any two factors might have been taken; but ſince the leſſer of the two ſquares ſought muſt be greater than bb, the queſtion now turns upon this hinge, *viz.* *To reſolve the given difference* d *into two ſuch factors, that the ſquare of half the difference of thoſe factors may be greater than* bb. To reſolve this queſtion, let z and $\frac{d}{z}$ be the two factors ſought, ſince $z\times\frac{d}{z}=d$; then will the difference of theſe two factors be $z-\frac{d}{z}$ or $\frac{zz-d}{z}$; whence half the difference will be $\frac{zz-d}{2z}$; and it is the ſquare of this half difference

that

that muſt be greater than bb, that is, $\frac{z^4-2dz^2+d^2}{4z^2}$ muſt be greater than bb: treat this inequality like an equation, that is, multiply all by $4z^2$, and then $z^4-2dz^2+d^2$ muſt be greater than $4b^2z^2$, and $z^4-2dz^2-4b^2z^2+d^2$ muſt be greater than nothing: but $d+bb=aa$ by the ſuppoſition, and $2d+2b^2=2a^2$, and $2d+4b^2=2a^2+2b^2$; ſubſtitute then $2a^2+2b^2$ inſtead of $2d+4b^2$ in the quantity $z^4-2dz^2-4b^2z^2+d^2$, and you will have $z^4-2a^2z^2-2b^2z^2+d^2$ greater than nothing, and $z^4-2a^2z^2-2b^2z^2$ greater than $-dd$; but $d=a^2-b^2$, and $d^2=a^4-2a^2b^2+b^4$; whence $-d^2=-a^4+2a^2b^2-b^4$; ſubſtitute this laſt quantity inſtead of $-d^2$ in the foregoing account, and you will have $z^4-2a^2z^2-2b^2z^2$ greater than $-a^4+2a^2b^2-b^4$; proceed as in a quadratic equation, that is, ſince the ſecond term here is $-2a^2z^2-2b^2z^2$, half the coefficient whereof is $-a^2-b^2$, add the ſquare of this half coefficient to both ſides, and you will have $z^4-2a^2z^2-2b^2z^2+a^4+2a^2b^2+b^4$ greater than $4a^2b^2$; extract the ſquare root of both ſides, and you will have $z^2-a^2-b^2$ either greater than $+2ab$, or leſs than $-2ab$, by the foregoing lemma; therefore z^2 muſt either be greater than $a^2+2ab+b^2$, or leſs than $a^2-2ab+b^2$; if z^2 be greater than $a^2+2ab+b^2$, then z muſt be greater than $+a+b$, or leſs than $-a-b$, by the abovecited lemma; if z^2 be leſs than $a^2-2ab+b^2$, then z muſt be leſs than $a-b$, and greater than $b-a$, by the ſame lemma; therefore if z be a factor for our purpoſe, it muſt either be taken without the limits $+a+b$ and $-a-b$, or elſe it muſt be taken within the narrower limits $a-b$ and $b-a$; ſo that z muſt not be any number between $a+b$ and $a-b$, nor any number between $b-a$ and $-b-a$. The caſe is this; that though z was made to repreſent but one of the factors ſought, yet, properly ſpeaking, it repreſents them both, ſince the proceſs and the concluſion will be the ſame, whichſoever of the two factors you make z to ſtand for: now *If* z *one of the factors be taken without the larger limits* $+a+b$ *and* $-a-b$, *the other factor* $\frac{d}{z}$ *will fall within the narrower limits* $a-b$ *and* $b-a$; *and* vice verſa, *if* z *be taken within the narrower limits,* $\frac{d}{z}$ *will fall without the more diſtant limits*; of which I ſhall give the following demonſtration.

CASE I.

Let z be taken greater than $a+b$; then will $\frac{d}{z}$ be leſs than $\frac{d}{a+b}$, becauſe the former fraction hath the greater denominator; but $\overline{a+b}\times\overline{a-b}$ $=a^2$

$=a^2-b^2=d$; therefore $\frac{d}{a+b}=a-b$; but $\frac{d}{z}$ was found lefs than $\frac{d}{a+d}$, therefore $\frac{d}{z}$ will be lefs than $a-b$; and that $\frac{d}{z}$ is greater than $b-a$ is evident, fince the former is an affirmative quantity, and the latter a negative one.

CASE 2.

Let now $-z$ be a factor taken lefs than $-a-b$; then will the other factor be $-\frac{d}{z}$; and fince $-z$ is lefs than $-a-b$, $+z$ will be greater than $+a+b$, and $\frac{d}{z}$ will be lefs than $a-b$, as before, and $-\frac{d}{z}$ will ftill be lefs than $a-b$; but if $\frac{d}{z}$ be lefs than $a-b$, then $-\frac{d}{z}$ muft be greater than $b-a$; whence it appears, that if in any cafe z one of the factors be taken without the limits $+a+b$ and $-a-b$, the other factor $\frac{d}{z}$ will fall within the limits $a-b$ and $b-a$. *And thus being directed to proper factors, the fquares fought will eafily be found by making their fides equal to half the difference and half the fum of the factors made choice of.*

For an example, let it be required to find a number which being feverally added to 3 and 2 will make them both fquares: here $a^2=3$, $b^2=2$, $d=1$, $a=\sqrt{3}$, $b=\sqrt{2}$; therefore the higheft and loweft limits are $+\sqrt{3}+\sqrt{2}$ and $-\sqrt{3}-\sqrt{2}$, and the intermediate limits are $\sqrt{3}-\sqrt{2}$ and $\sqrt{2}-\sqrt{3}$: now to take a factor above the higheft limit, it may be obferved, that 2 is greater than $\sqrt{3}$, and greater ftill than $\sqrt{2}$; therefore 4 is greater than $\sqrt{3}+\sqrt{2}$; make then 4 one of the factors, and confequently $\frac{1}{4}$ the other; then will half the fum of thefe factors be $\frac{17}{8}$, and half their difference $\frac{15}{8}$, and the two fquares fought will be $\frac{289}{64}$ and $\frac{225}{64}$, whereof the former exceeds 3, the latter exceeds 2, and the difference is 1: make now $x+3=\frac{289}{64}$ or $x+2=\frac{225}{64}$, and either equation will give $x=\frac{97}{64}$; therefore $\frac{97}{64}$ is a number, which being feverally added to 3 and 2 will make them both fquares.

I have

I have dwelt the longer upon this problem, as it introduces a new way of arguing very useful in *Diophantus* in determining the limits of his positions, &c.

PROBLEM 8.

Having some relation to the 21st of the second book of Diophantus, *but being of a much greater extent.*

234. *To find a number, which being divided into any two parts whatever, the square of either part, together with a hundred times the other part shall make a square number.*

SOLUTION.

Put y for the number sought, and put x and $y-x$ for any two parts of that number, whereof the square of x and a hundred times $y-x$ both together are to make a square number: now the square of x is xx, and a hundred times $y-x$ is $100y-100x$; therefore $xx-100x+100y$ must be equated to a square; let $x-z$ or $z-x$ be the side of this square, be z what it will, and we shall have $xx-100x+100y=xx-2zx+zz$; cast away xx from both sides of the equation, and you will have $-100x+100y=-2zx+zz$: thus have we excluded xx out of the equation; and to do the same by x, to the end that it may not enter the conclusion, but that the value of y may be found without it, and x may be still left undetermined, let $2z=100$; then we shall have $z=50$, and $zz=2500$, and the equation will now stand thus, $-100x+100y=-100x+2500$; strike off $-100x$ from both sides, and you will have $100y=2500$, and $y=25$; therefore if $xx-100x+100y$ be the square of $z-x$ or $50-x$, y must be 25; and *e converso*, if y be 25, then $xx-100x+100y$ will be $xx-100x+2500$, which is the square of $50-x$, let x be what it will: but $xx-100x+2500$ is the square of x and a hundred times $25-x$ put together; therefore the square of x and a hundred times $25-x$ put together will always be a square number, let x be what it will; for it will always be the the square of $50-x$. Thus then we have found a number 25, which being divided into any two parts whatever, the square of either part together with a hundred times the other will make a square number: I say the square of either part; for as we have taken care in this solution not to confine x to any particular signification, but to leave it entirely unrestrained, x may with equal propriety represent either of the parts into which 25 is divided.

EXAMPLE.

Let the number 25 be divided into these two parts, 1 and 24: now the square of 1 is 1, and 100 times 24 is 2400, and 2401 is a square number

number whose side or root is 49. Let us again take these parts, but in a contrary order, and let 24 be the part to be squared: now the square of 24 is 576, and 100 times 1 is 100, and 576 and 100 taken together make 676 a square number, whose side is 26. And thus will this property be inseparable from the number 25, whether the two parts into which it is divided be integral or fractional; nay so inseparable is it, that it will subsist even when one of the parts is taken negative: as for example, let 25 be resolved into -5 and $+30$: now the square of -5 is $+25$, and 100 times 30 is 3000, and 3025 is a square number whose side is 55. If we take these parts in a contrary order, the square of 30 is 900, and a hundred times -5 is -500, and $900-500$ is 400 the square of 20.

N. B. This problem might have been proposed more generally thus: *To find a number which being divided into any two parts whatever, r times either part, together with the square of s times the other shall make a square number:* and if the foregoing process be repeated, and the operations be made in these more general terms, the number sought will be found at last to be $\frac{r}{4ss}$.

PROBLEM 9.

Being the 23d of the second book of Diophantus.

235. *To find two numbers such, that the square of each being added to the sum of them both may make a square number.*

SOLUTION.

Put x for one of the numbers sought, and it's square will be xx: now the square of $x+1$ is $xx+2x+1$; therefore $2x+1$ is such a number, as being added to xx the square of one of the numbers sought, will make a square; therefore if $2x+1$ be supposed to be the sum of the two numbers sought, one condition of the question will be answered. Let then $2x+1$ be the sum of the two numbers sought; and since x is named already for one of the numbers, the other number must be $x+1$, whose square is $xx+2x+1$; add to this square $2x+1$ the sum of the numbers, and you will have $xx+4x+2$, which, according to the other condition of the problem, must also be a square; let it then be a square whose side is $x-z$, and we shall have this equation, $xx+4x+2=xx-2zx+zz$; which equation, when resolved, gives $x=\frac{zz-2}{2z+4}$ for the less number; and if to this 1 be added, you will have the greater number sought; and z being left undetermined, any number, except

except unity, may be put for z. As for example, let $z=2$, and we shall have zz· $\frac{1}{4}$, and $\frac{1}{4}$ therefore $\frac{1}{4}$ and $\frac{5}{4}$ will answer the conditions of the problem. To try this, the sum of $\frac{1}{4}$ and $\frac{5}{4}$ is $\frac{6}{4}$ or $\frac{24}{16}$: now if to the square of $\frac{1}{4}$ (the lesser number) be added the sum of both, that is, if to $\frac{1}{16}$ be added $\frac{24}{16}$, you will have $\frac{25}{16}$ a square number: on the other hand, if to $\frac{25}{16}$ the square of $\frac{5}{4}$ (the greater number) be added the sum $\frac{24}{16}$, you will have $\frac{49}{16}$, which is also a square number.

PROBLEM 10.

Being the 31st of the second book of Diophantus.

236. *To find two numbers such, that if their sum be either added to or subtracted from the product of their multiplication, both the sum and the remainder shall be square numbers.*

SOLUTION.

Put x for one of the numbers sought, and then taking any two known numbers a and b, make a^2x+b^2x the other number sought, and the product of their multiplication will be $a^2x^2+b^2x^2$: now if $2abx^2$ be either added to or subtracted from this product, both the sum and the remainder will be square numbers; for the sum will be $a^2x^2+2abx^2+b^2x^2$, which is the square of $ax+bx$; and the remainder will be $a^2x^2-2abx^2+b^2x^2$, which is the square of $ax-bx$; and therefore if $2abx^2$ was the sum of the two numbers sought, both the conditions of the problem would be satisfied, let x be what it will: but as x is the less number, and a^2x+b^2x the greater, their sum will be a^2x+b^2x+x; therefore to make $2abx^2$ the sum of the two numbers sought, x must be restrained to such a value, that $2abx^2$ may be equal to a^2x+b^2x+x; be it so, and we shall have x, one of the numbers sought, equal to $\frac{a^2+b^2+1}{2ab}$; and if this number, when found, be multiplied into a^2+b^2, we shall have a^2x+b^2x the other number sought. As for example, since the numbers a and b are to be taken at pleasure, make $a=1$ and $b=2$, and you will have $\frac{a^2+b^2+1}{2ab}=\frac{6}{4}=\frac{3}{2}$ for the lesser of the two numbers sought;

and

and this multiplied into $a^2+b^2=5$, will give $\frac{15}{2}$ for the other number ſought. For a further proof of this, the product $\frac{3}{2}\times\frac{15}{2}$ is $\frac{45}{4}$, and the ſum $\frac{3}{2}+\frac{15}{2}$ is $\frac{18}{2}$ or $\frac{36}{4}$: now $\frac{45}{4}+\frac{36}{4}$, that is, $\frac{81}{4}$ is a ſquare number; and $\frac{45}{4}-\frac{36}{4}$, that is, $\frac{9}{4}$ is alſo a ſquare number.

PROBLEM II.

Being the 7th of the third book of Diophantus.

237. *To find three numbers ſuch, that not only the ſum of all three, but alſo the ſums of every two of them be ſquare numbers.*

This problem may be variouſly ſolved, according to the various arbitrary repreſentations that may be made of the ſums of the numbers ſought; but the moſt ſimple ſolution, as I take it, is as follows.

SOLUTION.

1ſt. Since the ſum of all the three numbers ſought is to be a ſquare, let it be the ſquare of $x+1$, and we ſhall have the ſum of all the three numbers equal to $xx+2x+1$.

2dly. Since the ſum of the firſt and ſecond numbers is to be a ſquare, let xx be that ſquare; and then if we ſubtract xx, the ſum of the firſt and ſecond numbers from $xx+2x+1$, the ſum of all three, we ſhall have $2x+1$ for the third number alone.

3dly. Again, ſince the ſum of the ſecond and third numbers is alſo to be a ſquare, let that be the ſquare of $x-1$; and then if from $xx-2x+1$, the ſum of the ſecond and third numbers, be ſubtracted $2x+1$, the third number alone, there will remain $xx-4x$ for the ſecond number alone.

4thly. Since, according to the ſecond ſtep, xx is the ſum of the firſt and ſecond numbers taken together, if from this ſum xx be ſubtracted the ſecond number $xx-4x$, there will remain $4x$ for the firſt number; and ſo the three numbers ſought will be thus repreſented:

1ſt, $4x$; 2d, $xx-4x$; 3d, $2x+1$.

5thly. But there is one condition of the problem ſtill unſatisfied, which is, that the ſum of the firſt and third numbers muſt, like the reſt, be a ſquare: now the third number is $2x+1$, and the firſt is $4x$, and their ſum is $6x+1$; therefore $6x+1$ muſt be equated to ſome ſquare: let

let us then aſſume ſome ſquare number greater than 25, for a reaſon hereafter to be given; and calling this aſſumed ſquare aa, let $6x+1$ be equal to aa, and we ſhall have $x=\frac{aa-1}{6}$; whence the following canon:

Taking any known ſquare number, as aa, *greater than twentyfive, make* $\frac{aa-1}{6}=x$, *and the three numbers ſought will be* $4x$, $xx-4x$ *and* $2x+1$.

But this ſolution is liable to two limitations: for firſt, x muſt be affirmative, becauſe the number $4x$ is ſo: ſecondly, $xx-4x$ muſt be affirmative, becauſe it is the ſecond number; that is, $xx-4x$ muſt be greater than nothing; therefore $x-4$ muſt be greater than nothing; therefore x muſt muſt be greater than 4; therefore $6x$ muſt be greater than 24, and $6x+1$ or aa muſt be greater than 25, as above.

EXAMPLE.

Let aa be taken equal to 121, which is the ſquare of 11; then will $\frac{aa-1}{6}$ or x be equal to 20; whence $4x$, or the firſt number, will be 80; $xx-4x$, or the ſecond number, will be 320; and $2x+1$, or the third number, will be 41: ſo the three numbers will be 80, 320 and 41. For firſt, the ſum of all three will be 441, the ſquare of 21 ſecondly, the ſum of the firſt and ſecond will be 400, the ſquare of 20 thirdly, the ſum of the ſecond and third will be 361, the ſquare of 19 and laſtly, the ſum of the third and firſt will be 121, the ſquare of 11.

PROBLEM 12.

Being the 12th of the third book of Diophantus.

238. *To find three numbers ſuch, that if to the product of every two of them a given number be added, the ſums may be all ſquare numbers.*

N. B. Though *Diophantus*'s ſolution of this problem be very artfully contrived, yet as it produces only fractional numbers, I prefer the following before it, which furniſhes as many anſwers in whole numbers as we pleaſe.

SOLUTION.

Let the three numbers ſought be a, b and c, and let e be the given number to be added to each product; then muſt $ab+e=\square$, $ac+e=\square$, and $bc+e=\square$: to effect which, it will be proper in the firſt place to enquire for two ſuch numbers a and b as will ſolve the firſt condition of the

the problem, thus: let nn be any ſquare number greater than e, and let $ab+e=nn$; then will $ab=nn-e$; whence it follows, that if the number $\overline{nn-e}$ be reſolved into any two factors whatever, theſe factors may be taken for a and b, and ſo we ſhall have two numbers a and b which will anſwer the firſt condition of the problem. But that we may not be obliged to tryals for proper values for a, b and n, let us aſſume any two numbers r and s, whereof r is greater than s, and whoſe ſquares are both greater than e, and let us make $n+r=a$, and $n-s=b$, that is, let $n+r$ and $n-s$ be the two factors into which the number $nn-e$ is ſuppoſed to be reſolved; then will $\overline{n+r}\times\overline{n-s}$ or $nn+rn-sn-rs=nn-e$; whence $rn-sn-rs=-e$, and $n=\frac{rs-e}{r-s}$, and

$r-s$ · 1 $r-s$, and $n-s$ or $b=\frac{ss-e}{r-s}$; therefore the values of a and b may be expreſſed two ways, to wit, either by $n+r$ and $n-s$, or (which amounts to the ſame thing) by $\frac{rr-e}{r-s}$ and $\frac{ss-e}{r-s}$.

Having thus ſecured two numbers a and b to anſwer the firſt condition of the problem, we are in the next place to enquire whether c will admit of ſuch a value, as being joined with thoſe of a and b above found will ſolve the other two conditions of the problem. Now to do this, from the number $ab+e$, which is already a ſquare, I ſubtract the number $ac+e$, which is to be made a ſquare, and find the difference to be $ab-ac$; and this difference being the product of a into $\overline{b-c}$, may be reſolved into thoſe two factors a and $\overline{b-c}$; of theſe two factors the ſemiſum is $\frac{a+b-c}{2}$, and the ſemidifference is $\frac{a-b+c}{2}$: again, from the ſame ſquare number $ab+e$ I ſubtract $bc+e$, which is to be a ſquare, and find the difference to be $ab-bc$, which is the product of b into $\overline{a-c}$: therefore it luckily happens, that the ſemiſum of theſe two factors, *viz.* $\frac{a+b-c}{2}$ is the ſame with the ſemiſum in the former caſe, but the ſemidifference is $\frac{a-b-c}{2}$; therefore if the ſquare of the ſemiſum $\frac{a+b-c}{2}$ can by any rational value of c be made equal to $ab+e$, from which both $ac+e$ and $bc+e$ were ſubtracted, the conſequence will be, that both $ac+e$ will become equal to the ſquare of the ſemidifference $\frac{a-b+c}{2}$,

and

and $bc+e$ will be equal to the ſquare of the ſemidifference $\frac{a-b-c}{2}$; and ſo both $ac+e$ and $bc+e$ will be ſquare numbers; all which is evident from article 231. Let us then ſuppoſe the ſquare of the ſemiſum $\frac{a+b-c}{2}$ to be equal to $ab+c$; and ſince it has been ſhewn above, that $ab+e$ is equal to nn, we ſhall have the ſquare of $\frac{a+b-c}{2}$ equal to nn; whence, taking the ſquare roots of both ſides, we have $\frac{a+b-c}{2} = \mp n$, and $c = a+b \pm 2n$: thus then we have two rational values of c, each of which, joined with the numbers a and b above found, will ſolve the problem, to wit, $a+b+2n$ and $a+b-2n$, the ſum of which two values of c is $2a+2b$; therefore if the leſſer value of c be found, and ſubtracted from $2a+2b$, the remainder will be the greater value: but to expreſs either of theſe values according to our preſent notation, we muſt have recourſe to our former notation, where a was made equal to $n+r$, and b to $n-s$; therefore $a+b = 2n+r-s$, and $a+b-2n = r-s$; therefore the leſſer value of c will always be equal to $r-s$; and therefore the greater value will always be the exceſs of $2a+2b$ above $r-s$; and ſo we ſhall have the following canon:

Take any two known numbers r *and* s, *whereof* r *is greater than* s, *and whoſe ſquares are both greater than* e; *make* $\frac{rr-e}{r-s} = a$, $\frac{ss-e}{r-s} = b$, *and make* r — s, *or the exceſs of* 2 a + 2 b *above* r — s, *equal to* c, *and you will have two values of* c, *either of which, joined with the two numbers* a *and* b *above found, will ſolve the problem.*

From this canon it plainly appears, that *If for* r *and* s *be taken any two numbers whoſe difference is unity, and whoſe ſquares are both greater than* e, *the values of* a *and* b, *and conſequently that of* c, *will be all whole numbers.* As for example, let e the given number to be added to each product be 3, and make $r=4$, and $s=3$; then will $\frac{rr-e}{r-s}$ or $a=13$, $\frac{ss-e}{r-s}$ or $b=6$, and $r-s$ or the leſſer value of c will be 1, and the greater value of c, to wit $2a+2b-1$, will be 37; whence we have two ſets of numbers that will ſolve the problem, to wit, either 1, 6 and 13, (for it matters not now in what order the numbers are taken,) or 6, 13 and 37, either of which ſets of numbers will ſolve the problem: for in the firſt ſet, to wit 1, 6 and 13, we have 1ſt $1 \times 6 + 3 = 9$ the ſquare of

17

of 3, 2dly $1\times13+3=16$ the ſquare of 4, and 3dly $6\times13+3=81$ the ſquare of 9; in the other ſet, to wit 6, 13 and 37, we have 1ſt $6\times13+3=81$ the ſquare of 9, 2dly $6\times37+3=225$ the ſquare of 15, and 3dly $13\times37+3=484$ the ſquare of 22.

Another example of the foregoing canon may be this: let 3 be the number to be added to each product as before; and make $r=5$, $s=3$, and you will have $\frac{rr-e}{r-s}$ or $a=11$, $\frac{ss-e}{r-s}$ or $b=3$, and $r-s$ or $c=2$; whence the other value of c will be $2a+2b-2=26$; and the two ſets of numbers that will ſolve the problem will be 2, 3 and 11, and 3, 11 and 26: in the firſt ſet, to wit 2, 3 and 11, we have 1ſt $2\times3+3=9$ the ſquare of 3, 2dly $2\times11+3=25$ the ſquare of 5, 3dly $3\times11+3=36$ the ſquare of 6; in the other ſet, to wit 3, 11 and 26, we have 1ſt $3\times11+3=36$ the ſquare of 6, 2dly $3\times26+3=81$ the ſquare of 9, and 3dly $11\times26+3=289$ the ſquare of 17.

SCHOLIUM 1.

If it were required *To find three ſuch numbers, that if from the product of every two of them a given number be ſubtracted, the remainders ſhall be all ſquares*, the foregoing canon with a very little alteration would ſuit this problem; I mean by changing the ſign of e wherever it is to be found, thus. In the foregoing canon a was taken equal to $\frac{rr-e}{r-s}$; therefore in this caſe a muſt be $\frac{rr+e}{r-s}$: there b was taken equal to $\frac{ss-e}{r-s}$; therefore here b muſt be $\frac{ss+e}{r-s}$: in the former caſe the greater value of c was the exceſs of $2a+2b$ above $r-s$, that is, the exceſs of $\frac{2rr+2ss-4e}{r-s}$ above $r-s$; therefore here it muſt be the exceſs of $\frac{2rr+2ss+4e}{r-s}$ above $r-s$, which, according to the ſenſe a and b are here taken in, will be the exceſs of $2a+2b$ above $r-s$. As for the leſſer value of c, to wit $r-s$, that will be the ſame in both caſes, becauſe the number e is not concerned in that expreſſion; therefore the canon for that problem is as follows:

Take any two known numbers r *and* s, *whereof* r *is greater than* s; *make* $\frac{rr+e}{r-s}=a$, $\frac{ss+e}{r-s}=b$, *and* $r-s$, *or the exceſs of* $2a+2b$ *above* $r-s$

equal to c, *and you will have three numbers* a, b *and* c, *which taken in any order will ſolve the problem.* As for example, let e, the number to be ſubtracted from each product, be 3; make $r=2$ and $s=1$, and you will have $\frac{rr+e}{r-s}$ or $a=7$, $\frac{ss+e}{r-s}$ or $b=4$, and $r-s$ or the leſſer value of c equal to 1, and $2a+2b-1$ or the greater value of c equal to 21; therefore the two ſets of numbers are 1, 4 and 7, and 4, 7 and 21: for in the ſet 1, 4 and 7 we have 1ſt $1\times4-3=1$ the ſquare of 1, 2dly $1\times7-3=4$ the ſquare of 2, and 3dly $4\times7-3=25$ the ſquare of 5; in the other ſet 4, 7 and 21 we have 1ſt $4\times7-3=25$ the ſquare 5, 2dly $4\times21-3=81$ the ſquare of 9, and 3dly $7\times21-3=144$ the ſquare of 12.

SCHOLIUM 2.

When the addititious number (which is to be added to the ſeveral products in order to make them ſquares) is itſelf a ſquare number, *Diophantus* has given us a very elegant ſolution of the caſe in the 20th queſtion of the fourth book of his Arithmetics, whereof I muſt here give ſome account, becauſe of the relation it hath to the thirteenth problem.

SOLUTION.

Let unity be the addititious number, (ſince, when a canon is calculated for the number 1, it may be eaſily made to ſuit any other ſquare number whatever, as will be ſhewn hereafter,) *and let it be required to find three numbers ſuch, that if an unit be added to the product of every two of them, the ſums may be all ſquares.* Here I put x for the firſt number ſought, and then aſſuming ſome known number as a, to the product ax I add 1, the ſquare root of the addititious number, and from the ſide $ax+1$ I raiſe the ſquare $aaxx+2ax+1$: now if unity be ſubtracted from this ſquare, there will remain $aaxx+2ax$; whence it follows, that if $aaxx+2ax$ be conſidered as the product of the firſt and ſecond numbers, one of the conditions of the problem will be anſwered; for then the product of the firſt and ſecond numbers, with an unit added to it, will be a ſquare number: let then $aaxx+2ax$ repreſent the product of the firſt and ſecond numbers multiplied together; and then ſince x is the firſt number, the ſecond will be $aax+2a$. Take now another known number as b, and from the ſide $bx+1$ raiſe the ſquare $bbxx+2bx+1$; and if from this ſquare an unit be ſubtracted, there will remain $bbxx+2bx$; therefore if $bbxx+2bx$ be conſidered as the product of the firſt and third numbers ſought, another condition of the problem will be anſwered: let then $bbxx+2bx$ repreſent the pro-

product of the firſt and third numbers ſought; and ſince by the ſuppoſition the firſt number is x, the third number muſt be repreſented by $bbx+2b$; and ſo the names of the three numbers will be 1ſt x, 2d $aax+2a$, 3d $bbx+2b$, and two of the three conditions of the problem will be ſatisfied. It remains now that we provide for the other condition, to wit, that the product of the ſecond and third numbers ſought may, after the addition of an unit, be a ſquare number: now the ſecond number was $aax+2a$, and the third was $bbx+2b$, and the product of theſe will be $aabbxx+\overline{2aab+2bba}\times x+4ab$; and if to this be added an unit, the ſum will be $aabbxx+\overline{2aab+2bba}\times x+\overline{4ab+1}$; but the ſecond term of this number, to wit, $\overline{2aab+2bba}\times x=2ab\times\overline{a+b}\times x$, and therefore if we make $a+b$ equal to c, the ſecond term of this ſum will be $2abcx$, and the whole ſum to be equated to a ſquare will be $aabbxx+2abcx+\overline{4ab+1}$: now it is certain that $aabbxx+2abcx+cc$ is a ſquare number whoſe ſide is $abx+c$; and therefore if $\overline{4ab+1}$, the laſt term of the ſum to be equated to a ſquare, was equal to cc, the laſt term of the abovementioned ſquare, then it is plain that the abovementioned ſum would be a ſquare number: let us then ſuppoſe cc or $aa+2ab+bb$ to be equal to $4ab+1$, and by tranſpoſition we ſhall have $aa-2ab+bb=1$, and conſequently $a-b$ or $b-a=1$. *If therefore* a *and* b *be ſo taken that their difference be unity or* 1, *the three numbers ſought will be 1ſt* x, *2d* aax+2a, *and 3d* bbx+2b; *and all the three conditions of the problem will be ſatisfied, whatever number the quantity* x *is made to ſtand for.* As for inſtance, make $a=1$, $b=2$, and the ſecond number $aax+2a$ will be $x+2$, and the third number $bbx+2b$ will be $4x+4$; and ſo the three numbers will be x, $x+2$, and $4x+4$: or if we make $x=1$, the three numbers will be 1, 3 and 8, whereof $1\times3+1=4$, $1\times8+1=9$, $3\times8+1=25$, which are all ſquare numbers.

If inſtead of unity any other ſquare number be made the addititious number, ſuppoſe cc, *then the three numbers above found, to wit* 1, 3 *and* 8, *or any other three numbers of the ſame kind that may be found by the ſame method, muſt be multiplied into* c *the ſquare root of the addititious number* cc, *and the products* 1c, 3c *and* 8c *will anſwer this caſe.* Thus $1c\times3c+cc=4cc$, $1c\times8c+cc=9cc$, and $3c\times8c+cc=25cc$: ſo that theſe laſt three ſums are nothing elſe but the former ſums multiplied into cc; and therefore if the former ſums be ſquare numbers, theſe laſt muſt be ſo too; for a ſquare number multiplying a ſquare number will always produce a ſquare.

PROBLEM 13.

Being the 21st of the fourth book of Diophantus.

239. *To find four numbers such, that if an unit, or any other square number be added to the product of every two of them, the sums may be all squares.*

SOLUTION.

Take three numbers a, b and c in arithmetical progression, so that their common difference be unity; and putting x for the first number sought, let $aax+2a$ represent the second, $bbx+2b$ the third, $ccx+2c$ the fourth; and by the second scholium in the foregoing article, the property of these four numbers will be, that the several products of the first and second, the first and third, and the first and fourth of these numbers, with an unit added to each will be all square numbers: it follows also from the same scholium, that the products of the second and third, and of the third and fourth, with an unit added to each will be square numbers; because the difference between a and b, as well as the difference between b and c, is unity: therefore if the product of the second and fourth numbers, with the addition of unity, can be made a square number by restraining x (which is yet indeterminate) to some particular value, then will all the conditions of the problem be satisfied.

But before we proceed any further upon this head, let us assign particular values to a, b and c. As for instance, make $a=1$, $b=2$, and $c=3$; then will the second number $aax+2a$ be equal to $x+2$, the third number $bbx+2b$ be equal to $4x+4$, and the fourth number $ccx+2c$ be equal to $9x+6$; and so the names of the four numbers will be x, $x+2$, $4x+4$ and $9x+6$. We are now to contrive that the product of the second and fourth of these numbers, with the addition of an unit, may be a square number: now the second number was $x+2$, and the fourth was $9x+6$, and the product of these two is $9xx+24x+12$, which with the addition of an unit is $9xx+24x+13$; therefore this last number must be equated to a square: now to find a proper side for such a square, since $3x$ is the square root of $9xx$ in the number above mentioned, if from $3x$ be subtracted 4, 5, 6, or any other number whose square exceeds 13, you will have a side to whose square the sum abovementioned may be equated; I shall here make the side $3x-11$, whose square is $9xx-66x+121$, and so feign the following equation, $9xx+24x+13=9xx-66x+121$; this equation being resolved gives $x=\frac{12}{10}$; whence $x+2$ or the second number equals $\frac{32}{10}$,

$4x+4$

$4x+4$ or the third number equals $\frac{88}{10}$, and $9x+6$ or the fourth number equals $\frac{168}{10}$: so the four numbers sought are $\frac{12}{10}$, $\frac{32}{10}$, $\frac{88}{10}$ and $\frac{168}{10}$. This is upon a supposition that the addititious number is unity; but if we make it any other square number, suppose 100, then the four numbers last mentioned must all be multiplied by 10 the square root of 100, and they will become the whole numbers 12, 32, 88 and 168. The proof is as follows:

$12\times 32+100=484$ the square of 22.
$12\times 88+100=1156$ the square of 34.
$12\times 168+100=2116$ the square of 46.
$32\times 88+100=2916$ the square of 54.
$32\times 168+100=5476$ the square of 74.
$88\times 168+100=14884$ the square of 122.

If the numbers a, b and c be taken equal to 2, 3 and 4 respectively, the four numbers sought will be x, $4x+4$, $9x+6$, and $16x+8$; the product of the second and fourth, with the addition of an unit, will be $64xx+96x+33$; and if this be made equal to the square of $8x-9$, we shall have the following equation, $64xx+96x+33=64xx-144x+81$; whence x the first number will be $\frac{2}{10}$, $4x+4$ the second number will be $\frac{48}{10}$, $9x+6$ the third number will be $\frac{78}{10}$, and $16x+8$ the fourth number will be $\frac{112}{10}$. This is supposing the addititious number to be an unit; but if we suppose it to be 100, and so multiply the four numbers abovementioned by 10, they will then be 2, 48, 78 and 112. See the proof:

$2\times 48+100=196$ the square of 14.
$2\times 78+100=256$ the square of 16.
$2\times 112+100=324$ the square of 18.
$48\times 78+100=3844$ the square of 62.
$48\times 112+100=5476$ the square of 74.
$78\times 112+100=8836$ the square of 94.

SCHOLIUM.

Monsieur *Bachet* in his comment upon the 12th question of the third book of *Diophantus*, from the 11th and 13th propositions of his second book of Porisms, analytically demonstrated in my solution of the foregoing

going twelfth problem, ſolves this problem two ways univerſally, whether the addititious number be a ſquare or not, and ſeems to value himſelf very much upon it, as having herein outdone even *Diophantus* himſelf. But to ſpeak freely my opinion of the matter, I think thoſe ſolutions would have appeared much more beautiful, had not the concluſions been ſo much embarraſſed with ſuch very high fractions: therefore I ſhall here paſs them over, and ſhall only produce a canon of my own, which though it does not ſolve the problem in whole numbers, unleſs the poſitions be ſo contrived before hand, yet it leads to fractions much more ſimple than thoſe of *Bachet*, as will eaſily appear to any one who ſhall think it worth his while to compare them with his in the place above-cited. The canon is as follows:

PROBLEM.

To find four numbers ſuch, that if to the product of every two of them any given number as c *be added, the ſums ſhall be all ſquare numbers.*

SOLUTION.

Aſſume any number as a, *whoſe ſquare is greater than* c; *ſubtract* $4aa-3c$ *from any ſquare number that is greater, ſuppoſe from* bb; *and then dividing the remainder by* $4a+2b$, *call the quotient* d; *make* $\frac{aa-c}{d}=e$, $d+e+2a=f$, *and* $3e+f+2a=g$; *then will the numbers* d, e, f, g *be ſuch as will anſwer the conditions of the queſtion.*

EXAMPLE.

1ſt. Let the given number to be added to the ſeveral products be 3; then will $c=3$.

2dly. Make $aa=4$; then will $4aa-3c=7$, and this ſubtracted from 9, a greater ſquare number, leaves a remainder of 2, which remainder being divided by $4a+2b$ or 14, gives $\frac{2}{14}$ or $\frac{1}{7}$ for the firſt number d.

3dly. According to this notation $a^2-c=1$, which being divided by d or $\frac{1}{7}$ quotes $e=7$.

4thly. $d+e=\frac{1}{7}+7=\frac{50}{7}$, and if to this be added $2a$ or $\frac{28}{7}$, we ſhall have $d+e+2a$, that is, $f=\frac{78}{7}$.

5thly.

5thly. $3e=21$ or $\frac{147}{7}$, and $f=\frac{78}{7}$, and $2a=\frac{28}{7}$; therefore $3e+f+2a$, that is, $g=\frac{253}{7}$: therefore the numbers d, e, f and g are $\frac{1}{7}$, $\frac{49}{7}$, $\frac{78}{7}$ and $\frac{253}{7}$ refpectively. See the proof:

$$\frac{1}{7}\times\frac{49}{7}+3\text{, that is, }\frac{1}{7}\times\frac{49}{7}+\frac{147}{49}=\frac{196}{49}\text{ the fquare of }\frac{14}{7}.$$
$$\frac{1}{7}\times\frac{78}{7}+3=\frac{225}{49}\text{ the fquare of }\frac{15}{7}.$$
$$\frac{1}{7}\times\frac{253}{7}+3=\frac{400}{49}\text{ the fquare of }\frac{20}{7}.$$
$$\frac{49}{7}\times\frac{78}{7}+3=\frac{3969}{49}\text{ the fquare of }\frac{63}{7}.$$
$$\frac{49}{7}\times\frac{253}{7}+3=\frac{12544}{49}\text{ the fquare of }\frac{112}{7}.$$
$$\frac{78}{7}\times\frac{253}{7}+3=\frac{19881}{49}\text{ the fquare of }\frac{141}{7}.$$

PROBLEM 14.

Being the 17th of the third book of Diophantus.

240. *It is required to find three numbers fuch, that the product of every two of them, together with their fum, may be a fquare number.*

This is the problem chiefly intended in this place: but becaufe a more general folution may be had at the fame expence as a particular one, and may be of fome ufe hereafter, as in the fcholium hereto annexed, and in the fifteenth problem following, I fhall propofe and folve the problem as follows:

It is required to find three numbers fuch, that the product of every two of them, together with t *times their fum, may make a fquare number.*

SOLUTION.

Let the three numbers fought be a, b and c, and the conditions of the problem may be expreffed by the three following equations;

$$ab+ta+tb=\square,$$
$$ac+ta+tc=\square,$$
$$bc+tb+tc=\square.$$

Now fince in this problem there are three unknown quantities, whereof only two enter every condition, this problem may be folved, in a great meafure,

measure, after the same manner as the twelfth, thus: assuming any known square number as n^2, make $ab+ta+tb=n^2$; then you will have $ab+ta+tb+t^2=n^2+t^2$; but $ab+ta+tb+t^2$ is nothing else but the product of $\overline{a+t}$ multiplied into $\overline{b+t}$; whence it follows, that if n^2+t^2 be resolved into any two factors, the lesser whereof is greater than t, these two factors may be taken for $a+t$ and $b+t$; and if t be subtracted from each, there will remain two numbers a and b, which will answer the first condition of the problem: but for more convenient values of a, b and n, take any two numbers r and s, whereof r is the greater, and make $n+r=a+t$, and $n-s=b+t$, that is, let $n+r$ and $n-s$ be the two factors into which the number n^2+t^2 is to be resolved; then will $\overline{n+r}\times\overline{n-s}$ or $n^2+rn-sn-rs=n^2+t^2$; therefore $rn-sn-rs=t^2$, and $n=\frac{rs+t^2}{r-s}$, and $n+r$ or $a+t=\frac{rs+t^2}{r-s}+\frac{r}{1}=\frac{r^2+t^2}{r-s}$: and for the same reason $n-s$ or $b+t=\frac{s^2+t^2}{r-s}$: therefore we have two ways of expressing the values of a and b, either by making a equal to $n+r-t$ and b equal to $n-s-t$, or (which amounts to the same thing) by making a equal to $\frac{r^2+t^2}{r-s}-t$, and b equal to $\frac{s^2+t^2}{r-s}-t$.

Having thus got two numbers a and b to answer the first condition of the problem, we are now in the next place to enquire, whether we cannot assign such a value to c, that c joined with the other two numbers a and b already found shall answer the other two conditions of the problem: now to try this, we have one number that is already a square, to wit, $ab+ta+tb$, and other two numbers which, if possible, are to be made squares, to wit, $ac+ta+tc$ and $bc+tb+tc$: to do this, from the first square $ab+ta+tb$ subtract the number $ac+ta+tc$, and the difference will be $ab-ac+tb-tc$, which difference is the product of $\overline{a+t}$ multiplied into $\overline{b-c}$; and the half sum of these two factors is $\frac{a+b-c+t}{2}$, but their half difference is $\frac{a-b+c+t}{2}$: again, from the same square $ab+ta+tb$ subtract the other number $bc+tb+tc$, and the difference will be $ab-bc+ta-tc$, which is the product of $\overline{a-c}$ into $\overline{b+t}$; but half the sum of these two factors is $\frac{a+b-c+t}{2}$, which is the same with the half sum in the former case; and half their difference is $\frac{a-b-c-t}{2}$;

whence

whence it appears by art. 231, that the ſolution of this problem will be poſſible: for if we make the ſquare of the ſemiſum $\frac{a+b-c+t}{2}$ equal to $ab+ta+tb$, the ſquare from which the other two were ſubtracted, and from thence get a rational value of c, we ſhall have not only $ac+ta+tc$ equal to the ſquare of the ſemidifference $\frac{a-b+c+t}{2}$, but alſo $bc+tb+tc$ equal to the ſquare of the ſemidifference $\frac{a-b-c-t}{2}$; and ſo not only $ac+ta+tc$, but alſo $bc+tb+tc$ will be a ſquare number. Let us then ſuppoſe the ſquare of $\frac{a+b-c+t}{2}$ equal to $ab+ta+tb$; and ſince $ab+ta+tb$ equals n^2 as above, we ſhall have the ſquare of $\frac{a+b-c+t}{2}$ equal to n^2, whence $\frac{a+b-c+t}{2}=\mp n$, and $c=a+b+t\pm 2n$. Here then we have two values of c, either of which, joined with the former numbers a and b, will ſolve the problem: the greater of theſe two values is $a+b+t+2n$, and the leſs $a+b+t-2n$; the ſum of theſe two values is $2a+2b+2t$, and their difference $4n$; therefore if the leſſer value of c be found, and either added to $4n$, or ſubtracted from $2a+2b+2t$, we ſhall either way have the greater value of c: but to find the leſſer value of c we muſt have recourſe to our former notation, where a was made equal to $n+r-t$, and b equal to $n-s-t$; therefore $a+b=2n+r-s-2t$, and $a+b+t=2n+r-s-t$, and $a+b+t-2n=r-s-t$; therefore in all problems of this kind, the leſſer value of c will be $r-s-t$, and therefore the greater value of c will either be $r-s-t+4n$, or (which is the ſame thing) the exceſs of $2a+2b+2t$ above $r-s-t$; and ſo we ſhall have the following canon.

Taking any two numbers r *and* s, *whereof* r *is the greater, make* $\frac{r^2+t^2}{r-s}-t=a$, $\frac{s^2+t^2}{r-s}-t=b$, *and make* $r-s-t$, *or the exceſs of* $2a+2b+2t$ *above* $r-s-t$ *equal to* c, *and you will have three numbers*, a, b *and* c, *which will anſwer the conditions of the problem.*

Here we may obſerve, 1ſt, that if t be ſuppoſed equal to 1, this problem will be changed into the problem firſt propoſed: 2dly, that if any two numbers whoſe difference is 1 be taken for r and s, the anſwer will all come out in whole numbers: 3dly, that in the caſe where $r-s$ equals 1 and t equals 1, the leſſer value of c, which is $r-s-t$ will always

ways be equal to nothing, and the greater value will always be equal to $2a+2b+2$: 4thly, that in the case where $r-s$ equals 1 and t equals 1, the foregoing canon will be changed into that which follows:

Taking any two numbers r *and* s *whose difference is* 1, *make* $r^2=a$, $s^2=b$, *and* o *or* $2a+2b+2=c$, *and you will have three numbers*, a, b *and* c, *which will answer the conditions of the problem.* As for example, supposing $t=1$, let $r=3$ and $s=2$; then we shall have $a=9$, $b=4$, $c=0$ or 28: therefore we have two sets of numbers (if I may call them so) that will answer the problem, to wit, 0, 4 and 9, and 4, 9 and 28; (for after the numbers are found, it matters not what order they are placed in:) and though the first of these two sets, *viz.* 0, 4 and 9 be of no consideration here, yet it may be of some consideration in another place: as to the other set, 4, 9 and 28, the proof is as follows:

$$4\times 9+4+9=49 \text{ the square of } 7.$$
$$4\times 28+4+28=144 \text{ the square of } 12.$$
$$9\times 28+9+28=289 \text{ the square of } 17.$$

SCHOLIUM.

If it be required *To find three numbers such, that if from the product of any two of them* t *times their sum be subtracted, the remainders shall be all square numbers*; this will easily be effected only by changing the sign of t in the several expressions of the foregoing problem: but here I must take notice, that as n is equal to $\frac{rs+t^2}{r-s}$, changing the sign of t will not affect n; for t^2 will be the same whether t be affirmative or negative. Let us now change the sign of t in the several expressions of the former problem thus: in the former problem we had $a=n+r-t$; therefore in this we have $a=n+r+t$: in the former problem we had $b=n-s-t$; therefore in this we have $b=n-s+t$: in the former problem we had $c=r-s-t$, or $r-s-t+4n$; therefore in this we have $c=r-s+t$, or $r-s+t+4n$: from all which it follows, that *If three numbers be found that will answer the fourteenth problem, and if to each of these numbers we add* 2t, *we shall have three numbers that will answer the conditions of this problem.* As for example, the numbers 4, 9 and 28 were found to answer the conditions of the former problem, when t was supposed equal to 1; add to each of these numbers $2t$, that is 2, and you will have 6, 11 and 30 to answer this problem. This appears by the proof; for

$$6\times 11-6-11=49 \text{ the square of } 7.$$
$$6\times 30-6-30=144 \text{ the square of } 12.$$
$$11\times 30-11-30=289 \text{ the square of } 17.$$

The other set of numbers for the foregoing problem were 0, 4 and 9; add 2 to each, and you will have the numbers 2, 6 and 11 for this; for

2×6

$2\times 6-2-6=4$ the square of 2.
$2\times 11-2-11=9$ the square of 3.
$6\times 11-6-11=49$ the square of 7.

Whence it follows that 2 will always be one number that will solve this problem, let r and s be what they will, provided their difference be unity.

PROBLEM 15.

Being the 14th of the third book of Diophantus.

241. *To find three numbers such, that if to the product of every two of them be added* t *times the third, the sums thence arising shall be all squares.*

SOLUTION.

Let the three numbers sought be a, b and c, and the conditions of the problem will be comprehended in these three equations;

1st, $ab+tc=\square$,
2d, $ac+tb=\square$,
3d, $bc+ta=\square$.

Subtract the second number from the first, that is, subtract $ac+tb$ from $ab+tc$, and the remainder will be $ab-ac+tc-tb$; but the remainder $ab-ac+tc-tb$ is the product of $\overline{a-t}$ multiplied into $\overline{b-c}$; and of these two factors $a-t$ and $b-c$ the semisum is $\frac{a+b-c-t}{2}$, and the semidifference $\frac{a-b+c-t}{2}$: again, subtract the third number from the first, that is, subtract $bc+ta$ from $ab+tc$, and the remainder will be $ab-bc+tc-ta$; but this remainder $ab-bc+tc-ta$ is the product of $\overline{a-c}$ into $\overline{b-t}$, of which two factors the semisum is $\frac{a+b-c-t}{2}$ as before, and the semidifference $\frac{a-b-c+t}{2}$. If now by supposing the square of the semisum (which in both cases is the same) equal to the first number, any rational value of c can be gained by such an equation, then it is plain that all the three numbers will be squares; the first for being equal to the square of the semisum, and the other two for being equal to the squares of the two semidifferences. But to contract the expression, in the semisum $\frac{a+b-c-t}{2}$ let us make $a+b-t=d$, and then the semi-

 sum

ſum of the factors will be $\frac{d-c}{2}$, and it's ſquare $\frac{cc-2cd+dd}{4}$: let us then make $\frac{cc-2cd+dd}{4}=ab+tc$, and we ſhall have $cc-2cd+dd=4ab+4ct$, and $cc-2cd-4ct+dd=4ab$, and $cc-2cd-4ct=4ab-dd$: now this is a quadratic equation, wherein c may be conſidered as the unknown quantity, and half the coefficient in the ſecond term is $-d-2t$; add the ſquare of this to both ſides, and you will have $cc-2cd-4ct+dd+4td+4tt=4ab+4td+4tt$; but $d=a+b-t$; ſubſtitute therefore this value of d into the ſecond ſide of the equation $4ab+4td+4tt$, and you will have $4ab+4ta+4tb$; therefore $cc-2cd-4ct+dd+4td+4tt=4ab+4ta+4tb$; therefore this equation will be inexplicable unleſs a and b can be ſo taken, as that $4ab+4ta+4tb$ ſhall be a ſquare number. Let the numbers a, b and n be taken the ſame here as in the 14th problem, and you will have (as there) $ab+ta+tb=n^2$, and $4ab+4ta+4tb=4n^2$, and we ſhall now have $cc-2cd-4ct+dd+4td+4tt=4n^2$; extract the ſquare root of both ſides, and you will have $c-d-2t=\pm 2n$; and $c=d+2t\pm 2n=a+b-t+2t\pm 2n=a+b+t\pm 2n$; but $a+b+t\pm 2n$ contained the two values of c in the 14th problem; therefore any three numbers that will anſwer the 14th problem will alſo anſwer this: therefore theſe two problems may now be joined into one, and the ſame anſwer will ſerve for both, as follows:

PROBLEM.

To find three numbers ſuch, that if to the product of every two of them be added either t *times their ſum, or* t *times the third number, the ſums thence ariſing ſhall be all ſquares.*

ANSWER.

Find three numbers from the canon delivered in the fourteenth problem or in the ſcholium, and theſe three numbers will ſolve this double problem. As for example, the numbers 4, 9 and 28 were found by the canon in the 14th problem, ſuppoſing $t=1$: I ſay then that the numbers 4, 9 and 28 are ſuch, that if to the product of every two of them be added either their ſum or the third number, the ſums thence ariſing ſhall be all ſquares: for

1ſt, $4\times 9+4+9=49$, and $4\times 9+28=64$;
2dly, $4\times 28+4+28=144$, and $4\times 28+9=121$;
laſtly, $9\times 28+9+28=289$, and $9\times 28+4=256$;

all which are ſquare numbers.

Again, in the ſcholium to the foregoing problem, where t was ſuppoſed equal to -1, the numbers found by that canon were 2, 6 and 11; therefore

therefore the numbers 2, 6 and 11 are ſuch, as if from the product of every two of them be ſubtracted either their ſum or the third number, the remainders ſhall be all ſquares: for

1ſt, $2\times 6-2-6=4$, and $2\times 6-11=1$;
2dly, $2\times 11-2-11=9$, and $2\times 11-6=16$;
laſtly, $6\times 11-6-11=49$, and $6\times 11-2=64$;

all which are ſquare numbers.

Theſe are very curious properties of the three numbers ſought; and what makes it more ſurprizing, is the infinite variety of anſwers theſe problems will admit of, and all in whole numbers. *Diophantus* has given us no demonſtration of the theorem in the fourteenth problem from which all theſe anſwers are derived, but rather refers to it as to a poriſm elſewhere by him demonſtrated: but theſe poriſms are now all loſt, or at leaſt not publiſhed. Monſieur *Bachet* indeed, in the ſixteenth and ſeventeenth propoſitions of his ſecond book of poriſms, has given us demonſtrations, ſuch as they are, of the theorems in the fourteenth problem and the ſcholium: but in the firſt place, he demonſtrates but one ſingle caſe of thoſe theorems, to wit, when $t=r-s$; and in the next place, the demonſtrations he gives us are only ſynthetical, and ſo abominably perplexed withal, that in each demonſtration he makes uſe of all the letters in the alphabet, except the letters *I* and *O*, ſingly to repreſent the quantities he has there occaſion for: nor has the matter been much mended ſince by our countreyman *Kerſey*. But I believe it may be reaſonably queſtioned, whether any one curious in theſe matters can be ſatisfied with ſuch ſort of demonſtrations; and therefore I have taken the liberty to treat theſe two laſt problems and the twelfth more in an analytical way, and ſubmit it to the judgement of others, whether in ſuch-like caſes this is not the more natural method to proceed in.

PROBLEM 16.

Being the twentieth of the third book of Diophantus.

242. *To find two numbers ſuch, that each and their ſum being ſeverally added to the product of their multiplication, the three numbers thence ariſing ſhall be all ſquares.*

SOLUTION.

Put x for one of the two numbers ſought; then ſince this number and the product of the multiplication of both, when added together, muſt make a ſquare, let this ſquare be xx; (and it might have been $4xx$, $9xx$, &c;) then will the product of the two numbers ſought be $xx-x$; and therefore

therefore, as x repreſents already one of the numbers ſought, the other muſt be $x-1$, and ſo one of the conditions of the problem will be fulfilled: again, if to $xx-x$, the product of their multiplication, be added the other number $x-1$, we ſhall have $xx-1$; and if to the ſame product $xx-x$ be added the ſum of the numbers $2x-1$, we ſhall have $xx+x-1$; and according to the other conditions of the problem, both theſe numbers, to wit, $xx-1$ and $xx+x-1$ are to be ſquare numbers. To effect this, ſubtract one of the numbers from the other, and the difference will be x: now the chief difficulty is, how to reſolve this difference x into two ſuch factors, that the ſquare of the ſemiſum of theſe factors being equated to the greater number, will give a rational value of x; for then both $xx+x-1$ and $xx-1$ will be true ſquares: let us then reſolve the difference x into two indeterminate factors, to wit, z and $\frac{x}{z}$, and their ſemiſum will be $\frac{x}{2z}+\frac{1}{2}z$, and the ſquare of this will be $\frac{x^2}{4z^2}+\frac{1}{2}x+\frac{1}{4}z^2=x^2+x-1$: now here it is eaſy to ſee, that this equation will be inexplicable, unleſs z be ſuch a number that $\frac{x^2}{4z^2}$ ſhall be equal to x^2; but if $\frac{x^2}{4z^2}$ be equal to x^2, then $\frac{x^2}{4z^2}$ being thrown off from one ſide of the equation, and it's equal x^2 from the other, there will remain a ſimple equation for determining the value of x: let then $\frac{x^2}{4z^2}=x^2$, and we ſhall have z, one of the factors, equal to $\frac{1}{2}$, and conſequently $\frac{x}{z}$, the other factor, equal to $\frac{x}{\frac{1}{2}}$ or $2x$; and ſo the factors here to be uſed are $2x$ and $\frac{1}{2}$.

This being diſcovered, let us now begin again, and inſtead of reſolving the difference x into the two factors z and $\frac{x}{z}$, let us now reſolve it into the two factors $\frac{1}{2}$ and $2x$, and their ſum will be $2x+\frac{1}{2}$, and their ſemiſum $x+\frac{1}{4}$, and the ſquare of this $xx+\frac{1}{2}x+\frac{1}{16}=xx+x-1$; throw away xx from both ſides, and you will have $\frac{1}{2}x+\frac{1}{16}=x-1$; whence x, one of the numbers ſought, equals $\frac{17}{8}$, and conſequently x

-1,

-1, the other number fought, equals $\frac{9}{8}$; therefore $\frac{9}{8}$ and $\frac{17}{8}$ are two fuch numbers as the problem requires; for the product of $\frac{9}{8}$ and $\frac{17}{8}$ is $\frac{153}{64}$, and their fum is $\frac{26}{8}$: now

$$\frac{153}{64}+\frac{9}{8} \text{ or } \frac{153+72}{64}=\frac{225}{64} \text{ the fquare of } \frac{15}{8}; \text{ and}$$
$$\frac{153}{64}+\frac{17}{8} \text{ or } \frac{153+136}{64}=\frac{289}{64} \text{ the fquare of } \frac{17}{8}; \text{ and laftly}$$
$$\frac{153}{64}+\frac{26}{8} \text{ or } \frac{153+208}{64}=\frac{361}{64} \text{ the fquare of } \frac{19}{8}.$$

PROBLEM 17.

Being the 4th of the fourth book of Diophantus.

243. *To find two numbers fuch, that if to the fquare of the firft number and to it's fide the fecond be added, there fhall arife the fquare of a third number and it's fide.*

SOLUTION.

Let x and y be the two numbers fought; then according to the condition of the problem, if y be added to xx and x, the firft fum will be a fquare, and the other fum will be the fide of that fquare; therefore $xx+y$ is a fquare, and $x+y$ the fide of that fquare; therefore the fquare of the latter will be equal to the former, that is, $xx+2xy+yy=xx+y$; therefore $2xy+yy=y$, and (dividing by y) $2x+y=1$, whence $y=1-2x$: therefore if for x be taken any fraction lefs than $\frac{1}{2}$, and y be taken equal to $1-2x$, you will have x and y the two numbers fought. As for example, let $x=\frac{2}{5}$, then will $1-2x$ or $y=\frac{1}{5}$: now if $x=\frac{2}{5}$, we fhall have $xx=\frac{4}{25}$; and we are to examine, whether if to $\frac{4}{25}$ and $\frac{2}{5}$ be added $\frac{1}{5}$, there will arife a fquare and it's fide: now $\frac{4}{25}+\frac{1}{5}=\frac{9}{25}$ a fquare number; and $\frac{2}{5}+\frac{1}{5}=\frac{3}{5}$ the fide of the fquare $\frac{9}{25}$.

N. B. The reafon why x muft be taken lefs than $\frac{1}{2}$ is, that $1-2x$ or y may be affirmative.

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PROBLEM 18.

Being the 5th of the fourth book of Diophantus.

244. *To find two numbers ſuch, that if to the ſquare of the firſt and it's ſide, the ſecond be ſeverally added, there ſhall ariſe two numbers, whereof the firſt is the ſide of the ſecond being a ſquare.*

N. B. This problem is the reverſe of the foregoing.

SOLUTION.

Let x and y be the two numbers ſought; and let n be the ſide of the ſquare ariſing from the addition of y to the ſquare of x, and the conditions of the problem furniſh the two following equations;

1ſt, $x^2+y=n$, and
2dly, $x+y=n^2$.

The firſt equation gives $y=n-x^2$, and the ſecond gives $y=n^2-x$; therefore $n-x^2=n^2-x$: but this is a quadratic equation from whence no rational value of x can be obtained, except in ſome chance caſes. We muſt therefore try other poſitions, and ſee whether we cannot find ſuch as will ſucceed better, thus: inſtead of n let nx repreſent the number ariſing from the addition of y to the ſquare of x, and then the equations will be

1ſt, $x^2+y=nx$, and
2dly, $x+y=n^2x^2$.

The firſt equation gives $y=nx-x^2$, and the ſecond gives $y=n^2x^2-x$; therefore $nx-x^2=n^2x^2-x$; divide all by x, and you will have $n-x=n^2x-1$, which is a ſimple equation, and gives $x=\frac{n+1}{n^2+1}$; whence we have the following canon:

Aſſume any number greater than unity, and call it n; *then make* $\frac{n+1}{n^2+1}=x$, *and* $x\times\overline{n-x}=y$, *and* x *and* y *will be the two numbers ſought.* As for example, let $n=2$, then we ſhall have $\frac{n+1}{n^2+1}$ or $x=\frac{3}{5}$, and $x\times\overline{n-x}$ or $y=\frac{3}{5}\times\frac{7}{5}=\frac{21}{25}$; therefore the number to be ſquared is $\frac{3}{5}$, and the number to be added is $\frac{21}{25}$; now the ſquare of $\frac{3}{5}$ is $\frac{9}{25}$: let us then try whether $\frac{21}{25}$ being added to $\frac{9}{25}$ and $\frac{3}{5}$, will make the former ſum the ſide

side of the latter being a square: now $\frac{21}{25}$ added to $\frac{9}{25}$ gives $\frac{30}{25}$ or $\frac{6}{5}$; and $\frac{21}{25}$ added to $\frac{3}{5}$ or $\frac{15}{25}$ gives $\frac{36}{25}$, which $\frac{36}{25}$ is a square number, and $\frac{6}{5}$ it's side.

N. B. The reason why n must be assumed greater than unity is, that y or $x \times \overline{n-x}$ may be affirmative: for if $x \times \overline{n-x}$ be affirmative, then $n-x$ must be affirmative, that is, $n-x$ must be greater than nothing, and n must be greater than x; but $x = \frac{n+1}{n^2+1}$; therefore n must be greater than $\frac{n+1}{n^2+1}$; multiply all by n^2+1, and then n^3+n will be greater than $n+1$; subtract n from both sides, and you will have n^3 greater than 1, and therefore n must be greater than 1.

PROBLEM 19.

Being the 13th of the fourth book of Diophantus, *applied to squares instead of cubes.*

245. *To find two numbers such, that the square of the first being added to the second may be equal to the square of the second when added to the first.*

SOLUTION.

Taking any two known numbers, to wit, a greater as a and a less as b, let ax and bx represent the two numbers sought, and you will have $a^2x^2+bx=b^2x^2+ax$ by the supposition; resolve this equation, and you will have $x=\frac{a-b}{a^2-b^2}$; divide both the numerator and denominator of this fraction by $a-b$, and you will have $x=\frac{1}{a+b}$; therefore $ax=\frac{a}{a+b}$ and $bx=\frac{b}{a+b}$: and since the sum of these two numbers will always be $\frac{a+b}{a+b}$ or 1, let the numbers a and b be what they will; we have the following canon:

Divide unity into any two parts whatsoever, and those two parts will have the property the problem requires. As for example, let the two parts

 be

be $\frac{2}{5}$ and $\frac{3}{5}$: now if to $\frac{4}{25}$ the ſquare of the firſt part, be added the ſecond part $\frac{3}{5}$ or $\frac{15}{25}$, you will have $\frac{19}{25}$: on the other hand, if to $\frac{9}{25}$ the ſquare of the ſecond part be added the firſt part $\frac{2}{5}$ or $\frac{10}{25}$, you will have $\frac{19}{25}$ as before. Univerſally, let the parts be x and $1-x$: now if to xx the ſquare of the firſt part, be added the ſecond part $1-x$, you will have $1-x+xx$: ſo alſo if to $1-2x+xx$ the ſquare of the ſecond part, be added the firſt part x, you will have $1-x+xx$ as before.

This problem might have been more myſteriouſly propoſed thus: *To find a number, which being divided into any two parts whatever, the ſquare of one part being added to the ſecond, ſhall be equal to the ſquare of the ſecond part when added to the firſt.*

PROBLEM 20.

Being the 14th of the fourth book of Diophantus.

246. *To find two numbers ſuch, that not only each number, but alſo their ſum and their difference, being increaſed by unity, ſhall be all ſquares.*

SOLUTION.

Taking any indeterminate quantity x, multiply it by any number greater than unity, ſuppoſe by 3, and to the product $3x$ joining unity, from the ſide $3x+1$ form the ſquare $9xx+6x+1$; then it is plain that if the firſt number ſought be called $9xx+6x$, the firſt condition of the problem will be anſwered, becauſe this number being increaſed by unity will be a ſquare. Let us now aſſume any other indeterminate ſquare as yy; and if $yy-1$ be made to repreſent the ſecond number ſought, then the ſecond condition of the problem will alſo be provided for. But according to this notation, the ſum of the two numbers ſought will be $9xx+6x+yy-1$; and ſince the third condition of the problem requires, that this ſum increaſed by unity ſhall be a ſquare, we ſhall have $9xx+6x+yy=\square$; call this ſquare zz; and then ſince $9xx+6x+yy=zz$, it is plain that yy and zz will be two ſquares whoſe difference is $9xx+6x$; and therefore *e converſo*, if we can find two ſquares whoſe difference is $9xx+6x$, we may then make the leſſer of thoſe two ſquares equal to yy, and the three firſt conditions of the problem will ſtill be ſafe: but this muſt be done by art. 231, to wit, by reſol-

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ving the difference $9xx+6x$ into two factors by whose mutual multiplication that difference $9xx+6x$ is produced, and then taking the square of half the difference of those factors for the lesser of the two squares sought. Let the factors be $9x+6$ and x, and their difference will be $8x+6$, and half their difference $4x+3$, whose square is $16xx+24x+9$; make therefore $yy=16xx+24x+9$, and you will have $yy-1$, the second number sought, equal to $16xx+24x+8$. Thus then we have got two indefinite representations to represent the two numbers sought, to wit, $9xx+6x$ and $16xx+24x+8$, and there remains but one condition of the problem to be satisfied, which is, that the difference of these two last numbers being increased by unity shall also be a square: subtract therefore the less number $9xx+6x$ from the greater $16xx+24x+8$, and the difference will be $7xx+18x+8$, which being increased by unity must be a square; therefore $7xx+18x+9$ must be made a square. Now it is easy to see, from the nature of the foregoing operation, that the last member of this number, which in the present case is 9, will in all cases be some square number; and therefore it will be easy to feign a side to whose square it may be equated: in the present case, as the last member is 9 whose square root is 3, the side of the feigned square may be $3x-3$, $4x-3$, $5x-3$, &c; let it be $3x-3$, whose square is $9xx-18x+9$, and we shall have $7xx+18x+9=9xx-18x+9$; resolve this equation and you will have $x=18$; whence $9xx+6x$ or $\overline{9x+6}\times x$ will be 3024, and $16xx+24x$ or $\overline{16x+24}\times x$ will be 5616, and consequently $16xx+24x+8$ will be 5624: therefore the two numbers sought will be 3024 and 5624, whose sum is 8648, and whose difference is 2600; and if to all these four numbers we add unity, we shall have 1st 3025 the square of 55, 2dly 5625 the square of 75, 3dly 8649 the square of 93, and lastly 2601 the square of 51.

N. B. That the indeterminate quantity x in the beginning of this solution must be multiplied by some number greater than unity is plain; because if unity be made the multiplicator, and the reasoning be carried on as before, x will be found equal to unity, and so $yy-1$, or the second number, will be found equal to nothing.

PROBLEM 21.

Being the 32d question of this sort in Kersey's *Algebra.*

247. *To find three numbers such, that if to the square of each be added the sum of the other two, the numbers thence arising shall be all squares.*

SOLUTION.

The square of $x+1$ is $xx+2x+1$; whence it follows, that if we call the first number x, the second $2x$, and the third 1, the first conditi-

on of the problem will be satisfied; for then xx, the square of the first, together with $2x+1$, the sum of the other two, will be $xx+2x+1$, a square number. But the square of the second number together with the first and third must also be a square, that is, $4xx+x+1$ must be a square number; and again the square of the third number together with the first and second must be a square, that is, $3x+1$, must be a square; therefore if we will satisfy the two remaining conditions of the problem, we must resolve this duplicate equality, to wit, $3x+1=\square$, and $4xx+x+1=\square$. This duplicate equality is of a different form from any we have yet met with; but it may however be easily resolved by the usual methods thus: we are at liberty which of the two quantities $3x+1$ and $4xx+x+1$ we shall make the less; subtract then $3x+1$ from $4xx+x+1$, and the difference will be $4xx-2x$; therefore by art. 231 we are to find two squares whose difference is $4xx-2x$; and then by equating $3x+1$ to the lesser of those two squares, or $4xx+x+1$ to the greater, we shall have in either case the same value of x. Now the difference $4xx-2x$ is the product of $\overline{4x-2}$ multiplied into x, and the difference between these two factors $4x-2$ and x is $3x-2$, and half their difference $\frac{3x}{2}-1$; therefore by art. 231 above quoted, $\frac{3x}{2}-1$ is the side of the lesser square sought: hence we have this equation, $3x+1=\frac{9}{4}xx-3x+1$; resolve this equation, and you will have $x=\frac{8}{3}$; therefore the first number is $\frac{8}{3}$, the second $\frac{16}{3}$ and the third 1; and these numbers will satisfy the conditions of the problem: for $\frac{64}{9}$ the square of the first, with $\frac{57}{9}$ the sum of the second and third, makes $\frac{121}{9}$ the square of $\frac{11}{3}$: again, $\frac{256}{9}$ the square of the second number, with $\frac{33}{9}$ the sum of the first and third, makes $\frac{289}{9}$ the square of $\frac{17}{3}$: and lastly, 1 the square of the third, with 8 the sum of the first and second, makes 9 the square of 3.

SCHOLIUM.

Another example of the sort of duplicate equality used in the solution of the last problem may be this: *Let it be required to find such a value of* x, *that both* $8x+4$ *and* $3xx+9$ *shall be square numbers.*

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This form differs something from the foregoing, because there the quantities in the places of 4 and 9, which I shall call the absolute terms, were equal; however, as 4 and 9 are square numbers, it will be easy to reduce this form to that in the manner following: multiply the first quantity $8x+4$ by 9 the absolute term of the second, and the product will be $72x+36$; multiply also the second quantity $3xx+9$ by 4 the absolute term of the first, and the product will be $12xx+36$: now it is certain that if $72x+36$ be a square number, $8x+4$ will be so too, because the latter quantity is nothing else but the quotient of the former divided by 9; and every one knows, that if a square be divided by a square, the quotient will be a square: after the same manner it is shewn, that if $12xx+36$ be a square number, $3xx+9$ must be so, the latter quantity being a fourth part of the former: setting aside therefore the former question, the question may now be put thus; *To find such a value of* x, *that both* $72x+36$ *and* $12xx+36$ *may be square numbers*; the quantities being now in the same form as in the last article. Now the difference of the two quantities to be equated to squares is $12xx-72x$; and the next question that arises is, into what two factors must this difference be resolved, that the square of half the difference of these factors may be equated to the number $72x+36$ by a simple equation? to determine this point, let zx be one of the factors, and the other will be $\frac{12xx-72x}{zx}$ or $\frac{12x-72}{z}$; therefore the difference of the factors will be $\frac{12x}{z}-zx-\frac{72}{z}$: now as the square of half this difference is to be equated to the number $72x+36$ by a simple equation, it is plain that 36 must also be an ingredient in the square of that half difference; whence it follows, that 6 the square root of 36 must enter the half difference of the factors, and 12 the whole difference; whence at last it follows, that in the quantity $\frac{12x}{z}-zx-\frac{72}{z}$, the last member $\frac{72}{z}$ must be equal to 12; whence we have $z=6$, and $zx=6x$; therefore $6x$ must be one of the factors into which the difference $12xx-72x$ must be resolved; therefore to find the other factor, I divide the whole resolvend $12xx-72x$ by $6x$, and the quotient is $2x-12$; and so the two factors desired will be $6x$ and $2x-12$, whose difference is $4x+12$, and half difference $2x+6$, the square whereof is $4xx+24x+36=72x+36$; resolve this equation, and you will have $x=12$; therefore if x be made equal to 12, both $8x+4$ and $3xx+9$ will be square numbers; for the former will be 100 the square of 10, and the latter 441 the square of 21.

LEMMA.

LEMMA.

248. *Let* $\frac{a}{dd}$, $\frac{b}{dd}$ *and* $\frac{c}{dd}$ *be three fractions, all having the same square number for their denominator; and let the nature of these fractions be such, that the sum of every two of them be a square number: I say then, that if the common denominator be dropped, the numerators will still retain the same property: thus if* $\frac{a+b}{dd}=\square$, $\frac{b+c}{dd}=\square$, *and* $\frac{c+a}{dd}=\square$, *we shall have* $a+b=\square$, $b+c=\square$, *and* $c+a=\square$. *I say further, that if* $\frac{a-b}{dd}$ *be to* $\frac{b-c}{dd}$ *in any ratio or proportion whatever,* $a-b$ *will be to* $b-c$ *in the same ratio.*

All these things are so very manifest that they need no demonstration.

LEMMA.

249. *Let there be three numbers,* a, b *and* c, *and three others answering to them, as* d, e *and* f, *and let the excess of* a *above* b *have the same proportion to the excess of* b *above* c *as* d—e *hath to* e—f: *I say then, that if* c *the last number in one set be equal to* f *the last number in the other, and if* b *the middle term of the former set be equal to* e *the middle term of the latter,* a *the first term in one set will be equal to* d *the first term in the other; and therefore if* d *and* e *be both square numbers,* a *and* b *must be so too.*

For since *ex hypothesi*, $a-b$ is to $b-c$ as $d-e$ is to $e-f$, and since $b=e$ and $c=f$, we shall have $b-c=e-f$, that is, the two consequents of the foregoing proportion will be equal, and therefore the antecedents must be so too, that is, $a-b=d-e$; subtract the equal quantities $-b$ and $-e$ from the two sides of the equation, and you will have $a=d$. *Q. E. D.*

EXAMPLE.

Let the numbers proposed be $8x+4$, $6x+4$ and 4; and let the numbers answering to them be 64, 49 and 4: here then it is plain, that $2x$ the excess of $8x+4$ above $6x+4$, is to $6x$ the excess of $6x+4$ above 4, as 15 the excess of 64 above 49, is to 45 the excess of 49 above 4; it is plain also that 4 is the last term of each set, and therefore if the middle term $6x+4$ in one set be made equal to 49 in the other, then $8x+4$ the first number in one set must of course be equal to 64 the first number in the other; and as 64 and 49 are both square numbers, it follows upon this supposition, that $8x+4$ and $6x+4$ will both be squares; I mean upon the supposition that $6x+4=49$: this is evident from the demonstration of the lemma, and will further appear upon tryal; for if $6x+4=49$, we shall have $x=7\frac{1}{2}$; and if $x=7\frac{1}{2}$, we shall have $8x+4=64$.

PRO-

PROBLEM 22.

Being the 45th of the fourth book of Diophantus.

250. *To find three numbers ſuch, that the exceſs of the greateſt above the middle number ſhall be to the exceſs of the middle number above the leaſt in a given ratio, ſuppoſe as 3 to 1; and moreover that the ſum of every two of theſe numbers ſhall be a ſquare.*

SOLUTION.

Since of the three numbers ſought the ſum of every two is to be a ſquare, let the middle and leaſt numbers when added together make 4; then it is plain that the middle number alone muſt be greater than 2; for if it was equal to or leſs than 2, this middle number would be equal to or leſs than the leaſt, which is abſurd. Let us then call the middle number $x+2$, and ſubtracting this from 4, the remainder $2-x$ will repreſent the leaſt number; and the exceſs of $x+2$ the middle number above $2-x$ the leaſt, will be $2x$; therefore the exceſs of the greateſt number above the mean will be $6x$; whence the greateſt number will be $7x+2$; therefore the names of the three numbers ſought will be $7x+2$, $x+2$ and $2-x$, and two conditions of the problem will be fulfilled; for the exceſs of the greateſt number above the mean will be to the exceſs of the mean above the leaſt as 3 to 1; and moreover the middle and leaſt numbers taken together will make a ſquare. But the greateſt and middle numbers, and alſo the greateſt and leaſt, when added together muſt make ſquares: now the greateſt and middle numbers are $7x+2$ and $x+2$, which when added together, make $8x+4$; and the greateſt and leaſt numbers are $7x+2$ and $2-x$, which when added together, make $6x+4$; therefore both $6x+4$ and $8x+4$ muſt be ſquare numbers. Let us now in the firſt place reſolve this duplicate equality according to the common method, and let us ſee what will be the conſequence: now the difference between the two quantities that are to be equated to ſquares, *viz.* $6x+4$ and $8x+4$ is $2x$; therefore we muſt find two ſquares whoſe difference is $2x$, and then find the value of x by equating $6x+4$ to the leſſer, or $8x+4$ to the greater of theſe ſquares; but this may be done by art. 231, to wit, by reſolving the difference $2x$ into two factors, half the difference whereof will be the ſide of the leſſer ſquare ſought; and what theſe factors muſt be, may be determined thus: in the quantities to be equated, to wit, $6x+4$ and $8x+4$, the numeral parts are 4 and 4, which are ſquare numbers whoſe ſides are 2 and 2; therefore 2 and 2 muſt be the numeral parts of the half ſum and half difference of the factors whoſe ſquares are to be equated to $8x+4$ and

$6x+4$,

$6x+4$, that 4 being ſtruck off on both ſides, may leave a ſimple equation for determining the value of x: but if 2 be the numeral part of half the difference of the factors, then 4 will be the numeral part of the whole difference, and conſequently 4 muſt be one of the factors into which the difference $2x$ is to be reſolved; and the other factor will be $\frac{2x}{4}$ or $\frac{x}{2}$; therefore the difference of the factors will be $\frac{x}{2}-4$, and the half difference $\frac{x}{4}-2$, whereof the ſquare is $\frac{xx}{16}-x+4=6x+4$; reſolve this equation, and you will have $x=112$; therefore 112 is the value of x which will make both $6x+4$ and $8x+4$ ſquare numbers; and this is further confirmed upon tryal; for if x be 112, $6x+4$ will be 676 the ſquare of 26, and $8x+4$ will be 900 the ſquare of 30.

But though this value of x will ſolve the duplicate equality $6x+4=\square$ and $8x+4=\square$, it will by no means ſolve the problem, upon the account of ſome reſtrictions the poſitions lie under, which neither are, nor could be taken notice of in the duplicate equality as here reſolved. As for inſtance, the leaſt number was repreſented by $2-x$; but if $x=112$, that leaſt number $2-x$ will be negative, contrary to the intent of the problem: again, the middle number was $x+2$; but if $x=112$, $x+2$ (that middle number) will be 114, whereas the middle and leaſt numbers together ought to make but 4. Upon this account it was that our Author (who never wanted a ſhift to extricate himſelf out of the greateſt difficulties) invented another way of reſolving this duplicate equality, whereby he was ſure to gain his point with reſpect to the limitations above-mentioned, and for the better underſtanding whereof, I have in ſome meaſure prepared the reader in the laſt article.

We are then to take notice that of the two quantities to be equated, $8x+4$ exceeds $6x+4$ by $2x$; and again $6x+4$ exceeds it's numeral part 4 by $6x$; and $2x$ is the third part of $6x$; therefore $8x+4$, $6x+4$ and 4 are three ſuch numbers, that the exceſs of the greateſt above the mean is a third part of the exceſs of the mean above the leaſt: if therefore we can find three ſquare numbers under proper reſtrictions, whereof the leaſt is 4, and whereof the exceſs of the greateſt above the mean is a third part of the exceſs of the mean above the leaſt, theſe three ſquares will anſwer to the numbers $8x+4$, $6x+4$ and 4, in ſuch a manner, that if $6x+4$ the middle number in one ſet be equated to the middle ſquare in the other, then $8x+4$ will of courſe be equal to the greateſt ſquare, and ſo we ſhall have both $6x+4$ and $8x+4$ ſquare numbers, as was demonſtrated in the laſt article. Our firſt buſineſs then muſt be to find three ſuch ſquares as are here deſcribed, whereof 4 muſt

be

be the least: here then since 4 is a square number, let the side of the middle square be $y+\sqrt{4}$ or $y+2$, and the middle square will be y^2+4y+4, whose excess above 4 the least square, is y^2+4y, and a third part of this is $\frac{y^2+4y}{3}$; therefore $\frac{y^2+4y}{3}$ is the excess of the greatest square above the middle; add then this excess $\frac{y^2+4y}{3}$ to the middle square y^2+4y+4, and the greatest square will be $\frac{4}{3}y^2+\frac{16}{3}y+4$; this quantity therefore must be made a square; but to avoid fractions, multiply the whole by 9, and the product will be $12y^2+48y+36$; and to gain a more simple quantity, divide this last by 4, and the quotient will be $3y^2+12y+9$; therefore if this quantity $3y^2+12y+9$ can be made a square, the other $\frac{4}{3}y^2+\frac{16}{3}y+4$ must of course be a square, because the former quantity being multiplied by 4 and divided by 9, both which are square numbers, will be reduced to the latter. We are therefore in the next place to find a square number equal to $3y^2+12y+9$; to do which it must be observed, that the numeral part of this quantity $3y^2+12y+9$ is 9, and therefore the square to which it is to be equated must also have 9 for it's numeral part, that the equation whereby y is determined may be a simple equation: but if 9 be the numeral part of the square, 3 must be the numeral part of the side of that square; therefore the whole side must be 3 with a certain number of y's subtracted from it; but what this number of y's must be, remains in the next place to be determined. Let then z be the coefficient of y in the side of the square, and the whole side will be $3-zy$, the square whereof is $z^2y^2-6zy+9=3y^2+12y+9$; resolve this equation, and you will have $y=\frac{6z+12}{z^2-3}$.

The value of y being thus obtained, let us now look back upon our first positions, and we shall find, that according to our first notation, the third number sought was represented by $2-x$; whence it plainly appears, that x must be less than 2, and consequently $6x+4$ must be less than 16; but $6x+4$ is hereafter to be equated to the middle square; therefore the middle square must be less than 16, and it's side $y+2$ must be less than 4, whence y must be less than 2; but y was just now found equal to $\frac{6z+12}{zz-3}$; therefore $\frac{6z+12}{zz-3}$ must be less than 2, or (which is the same thing) z must be

greater than $\frac{6z+12}{zz-3}$; multiply both fides by $zz-3$, and you will have $2zz-6$ greater than $6z+12$, and $2zz$ greater than $6z+18$, and zz greater than $3z+9$, and $zz-3z$ greater than 9; add the fquare of half the coefficient of the fecond term, to wit $\frac{9}{4}$, and $zz-3z+\frac{9}{4}$ muft be greater than $\frac{45}{4}$; make then $zz-3z+\frac{9}{4}=\frac{49}{4}$, and you will have $z-\frac{3}{2}=\frac{7}{2}$, and $z=5$; put now 5 inftead of z in the fide of the fquare to which the quantity $3yy+12y+9$ is to be equated, and the fide of that fquare will be $3-5y$; whence we fhall have the following equation, $25yy-30y+9=3yy+12y+9$; refolve this equation, and you will have $y=\frac{21}{11}$; whence $y+2$ the fide of the middle fquare will be $\frac{43}{11}$, and the middle fquare itfelf will be $\frac{1849}{121}$. Hence alfo may the greateft fquare be found; but we have no occafion for it: for fince it was the middle fquare to which $6x+4$ was to be equated, we have this equation for determining the value of x, to wit, $6x+4=\frac{1849}{121}$; whence $6x=\frac{1365}{121}$, and $x=\frac{1365}{726}$.

This being gained, let us now return again to our firft pofitions; and the greateft number fought, which was $7x+2$, will now be $\frac{11007}{726}$; the middle number, which was $x+2$, will be $\frac{2817}{726}$; and the leaft, which was $2-x$, will be $\frac{87}{726}$. Thefe are numbers which will anfwer the conditions of the problem; but if whole numbers be defired, we muft proceed a little further thus: 726, the common denominator of all thefe fractions, is no fquare number, but rather the product of the fquare number 121 multiplied by 6; therefore if all the numerators and the common denominator be divided by 6, the fractions will be reduced to a fquare denominator, and will be as follows, $\frac{1834\frac{1}{2}}{121}$, $\frac{469\frac{1}{2}}{121}$ and $\frac{14\frac{1}{2}}{121}$. To avoid fractions in the numerators, multiply all the numerators and the common denominator by 4; and as 4 is a fquare number, the common denominator will ftill be a fquare number, and the fractions will now

now be $\frac{7338}{484}$, $\frac{1878}{484}$ and $\frac{58}{484}$. These fractions will answer the conditions of the question according to our first positions: but if we would have the answer in whole numbers, we must then set aside our first positions, and drop the common denominator, and the numerators will still retain all the properties required in the problem, as I have demonstrated in the last article but one: the first number then is 7338, the second 1878, and the third 58. The proof is as follows:

1st, $7338 - 1878$ is to $1878 - 58$ as 5460 is to 1820, that is, as 3 to 1.
2dly, $7338 + 1878 = 9216$ the square of 96.
3dly, $1878 + 58 = 1936$ the square of 44.
lastly, $58 + 7338 = 7396$ the square of 86.

SCHOLIUM.

Monsieur *Bachet* in his comment upon the fortyfifth question of the fourth book of *Diophantus*, reduces all that sort of duplicate equality which is resolvable according to the method last described, thus:

Let it be required that $ax + c = \square$, *and also that* $bx + d = \square$, *where it matters not what signs the quantities* a, b, c, d *are affected with; and let* $ax + c$ *be reputed greater than* $bx + d$, *or at least let* a *be greater than* b: *make* $\frac{b}{a - b} = q$, *and then multiplying the excess of* $ax + c$ *above* $bx + d$ *by that number* q, *and subtracting the product from* $bx + d$, *call the remainder* r: *I say then, that if* r *or* $\frac{-q}{r}$ *be a square number, the duplicate equality will be explicable this last way, otherwise not.*

In both these cases two squares must be found out, whereof the excess of the greater above the less must be to the excess of the less above r *as* 1 *is to* q; *and* $bx + d$ *must be equated to the lesser square. In the first case, where* r *is a square number,* $y + \sqrt{r}$ *must be made the side of the lesser square; but in the second case, where* $\frac{-q}{r}$ *and not* r, *is supposed to be a square number,* yy *must represent the lesser square.* The two following examples will clear up this whole matter.

First then, let it be required that both $3x + 13$ and $x + 7$ be square numbers. Here $a = 3$, $b = 1$, $\frac{b}{a - b}$ or $q = \frac{1}{2}$; and the excess of $3x + 13$ above $x + 7$ is $2x + 6$, which being multiplied into q or $\frac{1}{2}$, gives $x + 3$, and this last subtracted from $x + 7$ leaves 4 for r; therefore in this

this case r is a square number, and the duplicate equality will be explicable by finding two squares whereof the excess of the greater above the less is to the excess of the less above 4 as 1 to q, that is, as 1 to $\frac{1}{2}$ or as 2 to 1. Call the side of this lesser square $y+2$, and the square itself will be $yy+4y+4$, whose excess above 4 is $yy+4y$; double this, and $2yy+8y$ will be the excess of the greater square above the less; therefore the greater square will be $3yy+12y+4$, that is, the quantity $3yy+12y+4$ must be equated to a square: let $2-3y$ or $3y-2$ be the side of this square, and we shall have $3yy+12y+4=9yy-12y+4$; whence $y=4$, and $y+2$, the side of the lesser square, equals 6; therefore the square itself will be 36, and we shall have $x+7=36$, whence $x=29$, and the duplicate equality is resolved: for if $x=29$, we shall have $x+7=36$ a square number, and $3x+13=100$ a square number.

Note. The resolution of *Diophantus* in the last problem is of this sort.

Again, let it be required that both $6x+25$ and $2x+3$ be square numbers. Here $a=6$, $b=2$, and $\frac{b}{a-b}$ or $q=\frac{1}{2}$: the excess of $6x+25$ above $2x+3$ is $4x+22$, which being multiplied by q or $\frac{1}{2}$ gives $2x+11$; this subtracted from $2x+3$ leaves -8 for r; therefore in this case r is no square, but $\frac{-q}{r}$ or $\frac{-\frac{1}{2}}{-8}$ or $\frac{+1}{16}$ is a square; and therefore the duplicate equality will be explicable by finding two squares, whereof the excess of the greater above the less is to the excess of the less above -8 as 2 to 1. Let then yy represent the lesser of these two squares, and it's excess above -8 will be $yy+8$; double this, and $2yy+16$ will be the excess of the greater square above the less; therefore the greater square will be $3yy+16$, that is, the quantity $3yy+16$ must be equated to a square: let the side of this square be $3y-4$, and we shall have $3yy+16=9yy-24y+16$; whence $y=4$, and yy the lesser square equals 16; therefore $2x+3=16$, and $x=6\frac{1}{2}$, which will solve the problem: for if x be $6\frac{1}{2}$, we shall have $2x+3=16$ a square number, and $6x+25=64$ a square number; and the excess of 64 above 16 is to the excess of 16 above -8 as 48 is to 24, that is, as 2 to 1.

A LEMMATICAL PROBLEM.

251. *To find two square numbers with a given difference, and such, that the lesser of them shall be greater than any assigned number.*

SOLU-

SOLUTION.

Let d *be the given difference of the squares sought, and let* c *be the number which the lesser of the two squares is to exceed*; then if the difference d be resolved into two factors z and $\frac{d}{z}$, half the difference of these factors, to wit $\frac{zz-d}{2z}$ will be the side of the lesser square sought by art. 231; and therefore the square of this half difference must be greater than c, that is, $\frac{z^4-2dz^2+d^2}{4zz}$ must be greater than c; therefore $z^4-2dz^2+d^2$ must be greater than $4cz^2$; therefore $z^4-4cz^2-2dz^2+dd$ must be greater than nothing: make $z^4-4cz^2-2dz^2$ equal to or greater than nothing, and then *a fortiori*, $z^4-4cz^2-2dz^2+d^2$ will be greater than nothing; but if $z^4-4cz^2-2dz^2$ be equal to or greater than nothing, then $z^2-4c-2d$ will be equal to or greater than nothing, and z^2 will be equal to or greater than $4c+2d$: therefore *If any square number be taken equal to or greater than* $4c+2d$, *and if the side of that square be made one of the two factors into which the difference* d *is resolved, we shall by the help of these two factors, not only find two squares whose difference is* d, *but also such, that the lesser of them must be greater than* c. As for example, let it be required to find two square numbers whose difference shall be 21, and such, that the lesser of them shall be greater than 84. Here then $c=84$, $d=21$, $4c+2d=378$, and a square number greater than this, though not the next greater, is 441, whose side is 21; therefore 21 may be made one of the factors; and if so, then since the difference is 21, the other factor will be $\frac{21}{21}$ or 1; and therefore the two factors will be 21 and 1, whose half difference will be 10; and it's square 100: therefore the two squares sought are 100 and 121, whose difference is 21, and the lesser square 100 is greater than 84.

N. B. If c be taken equal to $4d$, as in the following problem, we shall have $4c+2d=18d$; therefore in this case, if any square number be taken equal to or greater than $18d$, the side of that square will be a proper factor.

PROBLEM 23.

Being the 2d of the fifth book of Diophantus.

252. *To find three numbers in continual proportion, and such, that each with a certain given number added to it shall be a square.*

SOLU-

SOLUTION.

Two ſquare numbers will always admit of a rational mean proportional between them, which mean proportional will be the product of their ſides: thus aa and xx will have ax for a mean proportional between them, that is, aa, ax and xx will be continual proportionals; for aa will be to ax as ax is to xx.

This being conſidered, let b be the given number, which being ſeverally added to the three numbers ſought, will make them all ſquares; and ſuppoſing two ſquare numbers to be found out whoſe difference is b, let aa be the leſſer of thoſe ſquares; then it is plain, that if aa be made one of the three numbers ſought, one of the conditions of the problem will be ſatisfied, for $aa+b$ is a ſquare number *ex conſtructione*: make then aa one of the extremes, and xx the other, and a middle proportional between them will be ax, as above; ſo the names of the three numbers ſought are aa, ax and xx; and as theſe three numbers are in continual proportion, another condition of the problem will be ſatiſfied. But there are two conditions ſtill remaining unanſwered; for according to the problem, both $ax+b$ and $xx+b$ muſt be ſquare numbers: now the difference of theſe two numbers $ax+b$ and $xx+b$ is $xx-ax$, ſince it matters not which is made the leſs; therefore if we find two ſquare numbers whoſe difference is $xx-ax$, and equate $ax+b$ to the leſſer ſquare, or $xx+b$ to the greater, we ſhall in either caſe be able to find out the reſt of the three numbers ſought. To effect this, reſolve the difference $xx-ax$ into the two factors x and $x-a$, and the difference of the factors will be a, and half their difference $\frac{a}{2}$, whoſe ſquare is $\frac{aa}{4}$; therefore by art. 231, $\frac{aa}{4}$ will be the leſſer of the two ſquares ſought, and we ſhall have $ax+b=\frac{1}{4}aa$; whence ax, the middle term of the three proportionals ſought, will be $\frac{1}{4}aa-b$. Thus having aa for one of the extremes, and $\frac{1}{4}aa-b$ for the middle of the three numbers ſought, the other extreme will be eaſily found by the rule of proportion, that is, by ſaying, As the known extreme is to the middle term, ſo is that middle term to the other extreme. But here we are to take notice, that to have the middle term (which is $\frac{1}{4}aa-b$) affirmative,

affirmative, $\frac{1}{4}aa$ muſt be greater than b, or (which comes to the ſame thing) aa muſt be greater than $4b$; whence we have the following canon:

Let b *be the given number, which being ſeverally added to the three numbers ſought, will make them all ſquares; then by the foregoing lemma find out two ſquare numbers whoſe difference is* b, *and ſuch, that the leſſer of them may be greater than* 4b; *call the leſſer of theſe two ſquares* a^2; *then if* a^2 *be made one of the extremes, and* $\frac{1}{4}a^2 - b$ *the middle term of the three numbers ſought, the other will be found by the rule of proportion.* As for example, let it be required to find three numbers in continual proportion, and ſuch, that each with 21 added may be a ſquare. Here $b=21$, and $4b=84$; therefore by the laſt article I find out two ſquare numbers whoſe difference is 21, and ſuch, that the leſſer of them ſhall be greater than 84: and out of an infinite choice I take the ſquares 100 and 121, as being whole numbers. Here then a^2, one of the extremes, is 100, and $\frac{1}{4}a^2-b$, the middle term, is 4; therefore the other extreme muſt be $\frac{16}{100}$, becauſe as 100 is to 4 ſo is 4 to $\frac{16}{100}$; and theſe numbers will anſwer the conditions of the queſtion: for

1ſt, They are in continual proportion *ex conſtructione.*

2dly, If 21 be added to 100, the ſum will be 121 the ſquare of 11.

3dly, If 21 be added to 4, the ſum will be 25 the ſquare of 5. And laſtly, If 21 be added to $\frac{16}{100}$, the ſum will be $\frac{2116}{100}$ the ſquare of $\frac{46}{10}$.

LEMMA. (See Plate I. Fig. 1.)

253. *If* ABC *be a plain triangle, whoſe two ſides* AB *and* BC *comprehend between them an angle of one hundred and twenty degrees, and are given; I ſay then that the ſquare of the third ſide* AC *may be had by adding the rectangle or product of the two ſides* AB *and* BC *to the ſum of their ſquares.*

But if ADC *be a plain triangle whereof the two ſides* AD *and* DC *comprehend between them an angle of ſixty degrees; I ſay then that the ſquare of the third ſide* AC *may be had by ſubtracting the rectangle or product of the two ſides* AD *and* DC *from the ſum of their ſquares.*

We are to prove, that $AB^2+AB\times BC+BC^2=AC^2$, and that $AD^2-AD\times DC+DC^2=AC^2$.

Produce AB out from B to D, ſo that BD may be equal to BC, and draw CD. Then ſince the angle ABC is an angle of one hundred and twenty degrees, the other angle CBD muſt be ſixty degrees, becauſe both toge-

together muſt make two right angles; therefore in the triangle BCD, the two angles BCD and D muſt both together make one hundred and twenty degrees; but thoſe two angles BCD and D are equal to each other, becauſe the ſides BC and BD are equal; therefore the triangle BCD is equiangular and conſequently equilateral. Make then $AB=a$, BC or CD or $BD=b$; and drawing the perpendicular CE, you will have $BE=\frac{1}{2}b$, and $BE^2=\frac{1}{4}bb$: but by the Pythagoric theorem $EC^2+EB^2=BC^2$, that is, $EC^2+\frac{1}{4}bb=bb$; therefore $EC^2=\frac{3}{4}bb$: but AE or $AB+BE=a+\frac{1}{2}b$, and $AE^2=aa+ab+\frac{1}{4}bb$; therefore AC^2, or $AE^2+EC^2=aa+ab+\frac{1}{4}bb+\frac{3}{4}bb=aa+ab+bb=AB^2+AB\times BC+BC^2$. Q. E. D.

To demonſtrate the other part, make now AD, and not AB, equal to a, and all things ſtanding as before, we ſhall now have $AE=a-\frac{1}{2}b$, and $AE^2=aa-ab+\frac{1}{4}bb$; whence we ſhall have AC^2, or $AE^2+EC^2=aa-ab+bb=AD^2-AD\times DC+DC^2$. Q. E. D.

COROLLARY 1.

If AB, BC and AC be three ſides of a triangle, whereof the ſides AB and BC contain an angle of one hundred and twenty degrees, and if DC be taken equal to BC, and AD equal to $AB+BD$ or $AB+BC$; then AD, DC and AC will be three ſides of a triangle, whereof AD and DC will contain an angle of ſixty degrees: or thus; *If* a, b *and* c *be three ſides of a triangle, whereof* a *and* b *contain an angle of one hundred and twenty degrees; then* b, a+b *and* c *will be three ſides of a triangle, whereof the two ſides* b *and* a+b *will contain an angle of ſixty degrees: and for a like reaſon,* a, a+b *and* c *will be three ſides of a triangle, whereof* a, *and* a+b *contain an angle of* 60 *degrees*; as is evident by producing the ſide CB to F, ſo that BF may be equal to BA, and joining AF; for then the three ſides of the triangle AFC will be a, $a+b$ and c.

COROLLARY 2.

If a, b *and* c *repreſent the three ſides of a triangle, whereof* a *and* b *make with each other an angle of* 120 *degrees, and if theſe three ſides be multiplied or divided by any given number whatever, as* d: *I ſay then that the products or quotients will form a triangle ſimilar to the former, and havi the ſame property*; that is, if the products be ad, bd and cd, we ſh have $a^2d^2+abd^2+b^2d^2=c^2d^2$, becauſe $a^2+ab+b^2=c^2$ *ex hypotheſi.*

PROBLEM 24.

Being the 7th of the fifth book of Diophantus.

254. *To find in whole numbers three ſides of an obtuſe-angled triangle, whoſe obtuſe angle ſhall be a hundred and twenty degrees.*

SOLU-

SOLUTION.

Take any known number as b to represent either of the sides about the obtuse angle, and call the other side x; then will the square of the third side opposite to the obtuse angle be $xx+bx+bb$ by the last article; let $a-x$ represent that third side, whose square is $aa-2ax+xx$, and we shall have this equation, $aa-2ax+xx=xx+bx+bb$; whence $x=\frac{aa-bb}{2a+b}$; therefore the sides about the obtuse angle will be $\frac{aa-bb}{2a+b}$ and b, that is, when reduced to the same denomination, $\frac{aa-bb}{2a+b}$ and $\frac{2ab+bb}{2a+b}$; but if $x=\frac{aa-bb}{2a+b}$, we shall have $a-x$ the third side, equal to $\frac{a}{1}\,\frac{-aa+bb}{2a+b}=\frac{aa+ab+bb}{2a+b}$; therefore the three sides according to this notation, are 1st $\frac{aa-bb}{2a+b}$, 2dly $\frac{2ab+bb}{2a+b}$, and 3dly $\frac{aa+ab+bb}{2a+b}$; drop the common denominator, which is the same as multiplying all by that common denominator, and the numerators will form a triangle similar to the former, by the second corollary in the last article, whose sides are now expressed in integral terms, to wit, $aa-bb$, $2ab+bb$ and $aa+ab+bb$.

Make $a+b=s$, and we shall have $aa+2ab+bb=ss$, and $2ab+bb=ss-aa$; moreover, as $aa+2ab+bb=ss$, we shall have $aa+ab+bb=ss-ab$; whence we have the following canon:

Take any two numbers a *and* b, *whereof* a *is the greater, and whose sum is* s: *then will the two sides about the obtuse angle be* aa—bb *and* ss—aa, *and the third side will be* ss—ab. As for example, make $a=2$, $b=1$, and consequently $s=3$, and you will have $aa-bb=3$, $ss-aa=5$ and $ss-ab=7$; so the sides are 3, 5 and 7, which will answer the conditions of the problem: for the sum of the squares of 3 and 5 is 34, and the product of their multiplication is 15, and these two added together make 49 the square of the third side 7.

If we join the two sides 3 and 5 together, their sum is 8, and this sum, with either of the sides about the obtuse angle, together with the third side 7, will form a triangle, as 3, 8 and 7, or 5, 8 and 7, whose angle opposite to the common side 7 will be 60 degrees, as is demonstrated in the first corollary in the last article.

PROBLEM 25.

Being the 8th of the fifth book of Diophantus.

255. *To find in whole numbers three right-angled triangles having all the same area.*

N. B. By the area of a right-angled triangle I mean half the product of it's legs or sides about the right angle; and therefore whenever the area of a right-angled triangle is required, it is only necessary the legs should be known, the hypotenuse being out of the question.

CANON.

Find by the help of the foregoing problem three whole numbers, a, b *and* c, *representing the three sides of a triangle, having opposite to* c *an angle of a hundred and twenty degrees: then by the help of these numbers form three right-angled triangles, to wit, the first by the help of the numbers* a *and* c, *the second by the numbers* b *and* c, *and the third by the numbers* $\overline{a+b}$ *and* c: *I say that these right-angled triangles will be such as the problem requires, having all the same area.* For

1st. In the triangle formed from the numbers a and c, one of the legs will be the difference of the squares of those numbers, to wit, $cc-aa$, and the other leg will be their double product, to wit, $2ac$, (see art. 12;) therefore the area of this triangle will be $\overline{cc-aa}\times ac$; but by art. 253, $cc=aa+ab+bb$; therefore $cc-aa=ab+bb=\overline{a+b}\times b$; therefore the area of the first triangle is $\overline{a+b}\times b\times ac=abc\times\overline{a+b}$.

2dly. In like manner, the area of the second triangle, formed from the numbers b and c, will be found to be $\overline{cc-bb}\times bc$; but $cc=aa+ab+bb$ as above, and $cc-bb=aa+ab=\overline{a+b}\times a$; therefore the area of the second triangle is $\overline{a+b}\times a\times bc=abc\times\overline{a+b}$, the same with the area of the first triangle.

3dly. The third triangle is formed from the numbers $\overline{a+b}$ and c, and therefore one of it's legs will be the difference of the squares of those numbers, and the other will be $\overline{a+b}\times 2c$; but $\overline{a+b}$ is greater than c; for a, b and c represent the three sides of a triangle, and in every triangle any two sides are greater than the third; therefore the difference of the squares of $a+b$ and c is $aa+2ab+bb-cc$; substitute instead of cc it's value $aa+ab+bb$, and you will have $aa+2ab+bb-cc=ab$; therefore ab will be one of the legs of the third triangle; and since the other

other leg is $\overline{a+b}\times 2c$ (as was obferved before,) the area of the third triangle will be $ab\times\overline{a+b}\times c = abc\times\overline{a+b}$; therefore all thefe three triangles have the fame area, to wit, $abc\times\overline{a+b}$. *Q. E. D.*

EXAMPLE.

In the example to the foregoing problem, the numbers 3, 5 and 7 reprefented three fides of a triangle whofe angle oppofite to 7 was one hundred and twenty degrees: form therefore three right-angled triangles, the firft from the numbers 3 and 7, the fecond from the numbers 5 and 7, and the third from the numbers 8 and 7 (8 being the fum of 3 and 5;) and the firft formed from the numbers 3 and 7 will have 40 and 42 for it's legs, and confequently 840 for it's area; the fecond formed from 5 and 7 will have 24 and 70 for it's legs, and confequently 840 for it's area; the laft triangle formed from 8 and 7 will have 15 and 112 for it's legs, and confequently 840 for it's area.

I fhall take this occafion to advertife the reader once for all, that *Fermat* in his new invented method of refolving duplicate equality, has carried this problem and feveral others much further than I have done; but his numbers running exceffively high, render his conclufions lefs agreeable to an elegant tafte, efpecially to one that wants rather to be acquainted with thefe matters than to fee the fubject quite exhaufted.

PROBLEM 26.

Being the 9th of the fifth book of Diophantus.

256. *To find three numbers fuch, that whether their fum be added to or fubtracted from the fquare of every particular number, the numbers thence arifing fhall be all fquares.*

SOLUTION.

I have demonftrated in art. 228, that in a right-angled triangle if the double product of the legs be either added to or fubtracted from the fquare of the hypotenufe, both the fum and remainder will be fquare numbers: but if half the product of the legs be the area of the triangle, the double product will be four times the area; fo that the above-quoted lemma may alfo be put thus: *If in a right-angled triangle four times the area be either added to or fubtracted from the fquare of the hypotenufe, both the fum and remainder will be fquare numbers.*

This being allowed, find (by the help of the foregoing problem) three right-angled triangles having all the fame area, and let their hypotenufes *be*

be a, b *and* c; *call also four times the common area* d, and from the foregoing lemma we shall have $a^2 \pm d = \square$, $b^2 \pm d = \square$, and $c^2 \pm d = \square$: let now these squares be all multiplied by some indeterminate square, as xx, and then we shall have $a^2x^2 \pm dx^2 = \square$, $b^2x^2 \pm dx^2 = \square$, and $c^2x^2 \pm dx^2 = \square$: this is universal, be the value of x what it will; but if we suppose the value of x to be such, that dx^2 is equal to the sum of all the three numbers ax, bx and cx, then we shall have three numbers ax, bx and cx of such a nature, that whether their sum be added to or subtracted from the square of every particular number, the numbers thence arising shall be all squares: let then $dx^2 = ax + bx + cx$, and we shall have $x = \frac{a+b+c}{d}$; or *If we make* a+b+c *(the sum of all the hypotenuses) equal to* s, we shall have $x = \frac{s}{d}$; whence ax, bx and cx *the three numbers sought will be* $\frac{as}{d}$, $\frac{bs}{d}$ *and* $\frac{cs}{d}$ *respectively.*

Now that these three numbers are such as the problem requires, may also be easily demonstrated synthetically thus: the sum of all the three numbers is $\overline{a+b+c} \times \frac{s}{d} = \frac{ss}{d}$, and the square of the first number $\frac{as}{d}$ is $\frac{a^2s^2}{d^2}$; therefore the sum of the three numbers being added to and subtracted from the square of the first gives $\frac{a^2s^2}{d^2} \pm \frac{s^2}{d} = \frac{a^2s^2 \pm ds^2}{d^2} = \overline{a^2 \pm d} \times \frac{ss}{d^2}$; but $a^2 \pm d$ are square numbers *ex hypothesi*; therefore $\overline{a^2 \pm d} \times \frac{s^2}{d^2}$ will be square numbers. And after the same manner it may be demonstrated, that if the sum of the three numbers be added to and subtracted from the square of the second number, and the square of the third, the sums and remainders shall be all square numbers.

EXAMPLE.

In the example to the foregoing problem we found three right-angled triangles having all the same area, to wit, 40, 42 and 58; 24, 70 and 74; and 15, 112 and 113, the common area being 840. Here then $a = 58$, $b = 74$, $c = 113$, $s = 245$, and $d = 840 \times 4 = 3360$; therefore $\frac{s}{d} = \frac{245}{3360} = \frac{7}{96}$; therefore $\frac{as}{d} = \frac{406}{96}$, $\frac{bs}{d} = \frac{518}{96}$, $\frac{cs}{d} = \frac{791}{96}$, which three last fractions are such as the problem requires, as any one may easily see that will be at the trouble of examining them.

PRO-

PROBLEM 27.

Being a case of the thirtieth of the fifth book of Diophantus.

257. *To find three square numbers such, that the sum of every two of them shall be a square.*

SOLUTION.

Put some known square number as dd *for one of the three squares sought*, and put xx and yy for the other two; then from the conditions of the problem we shall have these three equalities, to wit, $dd+xx=\square$, $dd+yy=\square$, and $xx+yy=\square$. Now to resolve the first of these equalities, to wit, $dd+xx=\square$, we must find two square numbers whose difference is dd, and then if we equate xx to the lesser of these two squares, we shall have $dd+xx=\square$; that is, by art. 231 we must resolve dd into two factors az and $\frac{dd}{az}$, where a signifies any determinate quantity, at least for the present till we shall be obliged to limit it; and then we must make the square of half the difference of these factors equal to xx; but half the difference of the factors az and $\frac{dd}{az}$ is $\frac{1}{2}az-\frac{1}{2}\frac{dd}{az}$, whether the expression be affirmative or negative it matters not; for in both cases we shall have $\frac{1}{4}aazz-\frac{1}{2}dd+\frac{\frac{1}{4}d^4}{a^2z^2}=xx$.

In like manner to fulfill the second condition of the problem, that is, to make $dd+yy=\square$, dd must be resolved into other two factors, bz and $\frac{dd}{bz}$, and the square of half the difference of these factors must be made equal to yy, and we shall have $\frac{1}{4}bbzz-\frac{1}{2}dd+\frac{\frac{1}{4}d^4}{bbzz}=yy$: put the values of xx and yy together, and you will have $xx+yy=\frac{1}{4}aazz+\frac{1}{4}bbzz-dd+\frac{\frac{1}{4}d^4}{aazz}+\frac{\frac{1}{4}d^4}{bbzz}$: make $aa+bb=cc$, that is, *let* a *and* b be now so limited as to *signify the two legs of any right-angled triangle whose hypotenuse is* c, and we shall have $\frac{1}{4}aazz+\frac{1}{4}bbzz=\frac{1}{4}cczz$; we shall have moreover $\frac{1}{aazz}+\frac{1}{bbzz}=\frac{aazz+bbzz}{aabbzzzz}=\frac{aa+bb}{aabbzz}$

$\frac{aa+bb}{aabbzz}=\frac{cc}{aabbzz}$; therefore we ſhall have $\frac{\frac{1}{4}d^4}{aazz}+\frac{\frac{1}{4}d^4}{bbzz}=\frac{\frac{1}{4}ccd^4}{aabbzz}$; therefore now $xx+yy=\frac{1}{4}cczz-dd+\frac{\frac{1}{4}ccd^4}{aabbzz}$: but by the laſt condition of problem, $xx+yy$ is to be a ſquare number; be it ſo, and let $\frac{1}{4}cczz$ be that ſquare number, and we ſhall have $\frac{1}{4}cczz=\frac{1}{4}cczz-dd+\frac{\frac{1}{4}ccd^4}{aabbzz}$; ſtrike off $\frac{1}{4}cczz$ from both ſides, and tranſpoſe $-dd$, and you will have $dd=\frac{\frac{1}{4}ccd^4}{aabbzz}$, and $1=\frac{\frac{1}{4}ccdd}{aabbzz}$, and $zz=\frac{\frac{1}{4}ccdd}{aabb}$, and $z=\frac{\frac{1}{2}cd}{ab}$, and $az=\frac{\frac{1}{2}cd}{b}$, and $\frac{dd}{az}=\frac{2bd}{c}$: but az and $\frac{dd}{az}$ were the firſt two factors into which dd was reſolved, and therefore half their difference will be x the ſide of the ſecond ſquare; but half the difference of $\frac{2bd}{c}$ and $\frac{\frac{1}{2}cd}{b}$ is $\frac{bd}{c}-\frac{\frac{1}{4}cd}{b}$; therefore $\frac{bd}{c}-\frac{\frac{1}{4}cd}{b}$, if affirmative, will be the ſide of the ſecond ſquare, if otherwiſe, it muſt be made affirmative by having it's ſign changed. Again, ſince $z=\frac{\frac{1}{2}cd}{ab}$, we ſhall have $bz=\frac{\frac{1}{2}cd}{a}$, and $\frac{dd}{bz}=\frac{2ad}{c}$: but bz and $\frac{dd}{bz}$ were the other two factors into which dd was reſolved, in order to expreſs (by half their difference) y the ſide of the third ſquare; therefore the ſide of the third ſquare will be $\frac{ad}{c}-\frac{\frac{1}{4}cd}{a}$; therefore *The ſides of the three ſquares ſought will be 1ſt* d, *2dly* $\frac{bd}{c}-\frac{\frac{1}{4}cd}{b}$, *3dly* $\frac{ad}{c}-\frac{\frac{1}{4}cd}{a}$. Now as we are entirely at liberty as to the value of the firſt ſquare dd, let us (to avoid fractions) make $d=4abc$, and we ſhall have $\frac{bd}{c}=4abb$, and $\frac{\frac{1}{4}cd}{b}=acc$, and the ſide of the ſecond ſquare will be $4abb-acc=a\times\overline{4bb-cc}$: in like manner, the ſide of the third ſquare will be $b\times\overline{4aa-cc}$: therefore now *The ſides of the three ſquares will be expreſſed all in whole numbers thus; 1ſt* $4abc$, *2dly* $a\times\overline{4bb-cc}$, *3dly* $b\times\overline{4aa-cc}$: *and thus by the help of any right-angled triangle whoſe ſides are whole numbers may this problem be reſolved.*

As

As for example, the numbers 3, 4 and 5 conſtitute a right-angled triangle; and if by the help of this triangle the problem is to be ſolved, we ſhall have $a=3$, $b=4$, $c=5$, $4abc=240$, $a\times\overline{4bb-cc}=117$, $b\times\overline{4aa-cc}=44$; and ſo the ſides of the three ſquares ſought (it matters not in what order they are taken) will be 44, 117 and 240; and therefore the ſquares themſelves will be 1936, 13689 and 57600: the ſum of the firſt and ſecond is 15625 the ſquare of 125; the ſum of the ſecond and third is 71289 the ſquare of 267; and the ſum of the third and firſt is 59536 the ſquare of 244.

PROBLEM 28.

Being the 32d of the fifth book of Diophantus.

258. *To find three ſquare numbers ſuch, that the ſum of their ſquares ſhall alſo be a ſquare number.*

SOLUTION.

Let a^2, b^2 and x^2 repreſent three ſuch ſquare numbers as the problem requires, and the ſum of their ſquares will be $a^4+b^4+x^4$; this the problem requires ſhall be a ſquare number: let it then be the ſquare of $c-x^2$, or rather of c^2-x^2 that the dimenſions may be alike, and we ſhall have $a^4+b^4+x^4=c^4-2ccxx+x^4$; reſolve this equation, and you will have $xx=\frac{c^4-b^4-a^4}{2cc}$: therefore if a, b and c be ſuch that $\frac{c^4-b^4-a^4}{2cc}$ be a ſquare number, the problem will be reſolvable, otherwiſe not.

Let $aa+bb=cc$, that is, let a and b be the two legs of any right-angled triangle whoſe hypotenuſe is c; and ſince $c^2=a^2+b^2$, we ſhall have $c^4=a^4+2a^2b^2+b^4$, and $c^4-b^4-a^4=2a^2b^2$, and $\frac{c^4-b^4-a^4}{2cc}$ or $xx=\frac{2aabb}{2cc}=\frac{aabb}{cc}$, which is a ſquare number: therefore *If* a, b *and* c *be three ſides of a right-angled triangle whoſe hypotenuſe is* c, *then* aa, bb *and* $\frac{\mathrm{aabb}}{\mathrm{cc}}$ *will be ſuch ſquares as the problem requires.*

As for example, the numbers 3, 4 and 5 conſtitute a right-angled triangle, where $aa=9$, $bb=16$, and $\frac{aabb}{cc}=\frac{144}{25}$: therefore 9, 16 and

$\frac{144}{25}$

$\frac{144}{25}$ are three ſuch ſquares as the problem requires; for the ſum of their ſquares is $81 + 256 + \frac{20736}{625} = \frac{231361}{625}$, whoſe ſquare root is $\frac{481}{25}$.

PROBLEM. 29.

Being the 33d of the fifth book of Diophantus.

259. This problem, as we have it in *Diophantus*, is contained in a Greek epigram, whether his own or not may I think be reaſonably queſtioned: this epigram Monſieur *Bachet* tranſlates as follows.

Drachmarum quinque, et drachmarum miſcuit octo
Quis choeas, famulis vina bibenda ſuis:
Pro cunctis pretium numerum praebens tetragonum,
Qui praefinitas ſuſcipiens monadas
Diverſum dat quadratum; ſed ſumma choarum
Illius exaequat conſtituitque latus.
Dic age quot choeas drachmarum comparat octo;
Drachmarum choeas, dic age, quinque, puer.

The purport whereof, except as to the money and meaſure there made uſe of, is this:

One buys in two ſorts of wines, a better ſort at the rate of eight pounds per *hogshead, and a worſe at five; the price of the whole amounted to a ſquare number of pounds, which with ſixty added made another ſquare number, whoſe ſide was the number of hogsheads of both ſorts put together. I demand how many hogsheads he bought in of each ſort, and what he paid for them.*

SOLUTION.

Put x for the number of hogſheads of both ſorts, and $xx - 60$ will repreſent, according to the problem, the whole number of pounds laid out; and this the problem requires ſhall be a ſquare: but before we can find a proper ſquare to which the number $xx - 60$ muſt be equated, we muſt make the following preparation.

Had the whole ſum $xx - 60$ been laid out in one ſort of wine only, ſuppoſe in the worſe ſort, the number of hogſheads bought in would have been $\frac{xx-60}{5}$; for if five pounds buy one hogſhead, $xx - 60$ will buy $\frac{xx-60}{5}$; and in this caſe, the number of hogſheads ſuppoſed to be

bought

bought in would have exceeded x the number of hogsheads actually bought in, for some whereof he paid eight pounds per hogshead; therefore $\frac{xx-60}{5}$ is greater than x, and $xx-60$ is greater than $5x$: by a like way of reasoning $xx-60$ must be less than $8x$, and therefore must consist between $5x$ and $8x$. Now if $xx-60$ be greater than $5x$, we shall have xx greater than $5x+60$, and $xx-5x$ greater than 60; compleat the square, by adding to both sides $\frac{25}{4}$, the square of half the coefficient of the second term, and you will have $xx-5x+\frac{25}{4}$ greater than $60+\frac{25}{4}$, that is, greater than $\frac{265}{4}$: let us suppose $xx-5x+\frac{25}{4}$ to be greater than $\frac{289}{4}$, and then the positions will not only be sufficiently guarded against any absurdity from that side, but also the limit will come out a rational number; for if $xx-5x+\frac{25}{4}$ be greater than $\frac{289}{4}$, $x-\frac{5}{2}$ will be greater than $\frac{17}{2}$, and x will be greater than $\frac{22}{2}$ or 11. Again, since $xx-60$ is less than $8x$, we shall have $xx-8x$ less than 60, and $xx-8x+16$ less than 76: let us suppose $xx-8x+16$ to be less than 64, and the positions will be also guarded on that side, and we shall have $x-4$ less than 8, and x less than 12; therefore the quantity x must be so limited as to be greater than 11 and less than 12: we might indeed, by a more nice extraction of the square root, have somewhat enlarged these limits, and have shewn that x may be any number between $10\frac{7}{10}$ and $12\frac{7}{10}$; but the limits above set down are exact enough for our purpose.

This being understood, let us now look back from whence we digressed, and try to make $xx-60$ a square number. Now to what square number soever $xx-60$ is equated, that we may have a simple equation, it is certain that x must be one part of the side of that square; but what must be the other part, so as to keep x within it's proper limits, remains still to be determined. Let then y be the other part, that is, let $xx-60$ be equated to a square whose side is $x-y$, and we shall have $xx-60=xx-2xy+yy$, and $x=\frac{yy+60}{2y}$: but x was found to be greater

 than

than 11 and leſs than 12; therefore the value of y muſt be ſuch that $\frac{yy+60}{2y}$ muſt be greater than 11 and leſs than 12. Firſt then, ſince $\frac{yy+60}{2y}$ is greater than 11, we ſhall have $yy+60$ greater than $22y$, and yy greater than $22y-60$, and $yy-22y$ greater than -60, and $yy-22y+121$ greater than 61: let $yy-22y+121$ be greater than 64, and we ſhall have $y-11$ greater than 8, and y greater than 19. Again, ſince $\frac{yy+60}{2y}$ is leſs than 12, we ſhall have $yy+60$ leſs than $24y$, and $yy-24y$ leſs than -60, and $yy-24y+144$ leſs than 84: let $yy-24y+144$ be leſs than 81, and we ſhall have $y-12$ leſs than 9, and y leſs than 21; therefore y muſt be greater than 19 and leſs than 21. Had we taken the more exact limits of x above-deſcribed, we ſhould have had more exact limits of y, to wit, $18\frac{1}{10}$ on one ſide and $22\frac{7}{10}$ on the other; but the limits here found, to wit, 19 and 21 are ſufficient. Since then y muſt be greater than 19, and leſs than 21, let us make y equal to 20, and ſo equate the quantity $xx-60$ to a ſquare whoſe ſide is $x-20$, and we ſhall have $xx-60=xx-40x+400$; whence $x=11\frac{1}{2}$, and is kept within its proper limits; therefore the whole quantity of wine bought in was 11 hogſheads and a half, or 23 half-hogſheads: but the ſquare of $\frac{23}{2}$ is $\frac{529}{4}$; and if from this be ſubtracted 60 or $\frac{240}{4}$, the remainder will be $\frac{289}{4}$, which is a ſquare number as the problem requires, and therefore properly repreſents the number of pounds laid out: but $\frac{289}{4}=72\frac{1}{4}$, and therefore the whole money laid out was $72\frac{1}{4}$ pounds.

Now to find the number of hogſheads of each ſort bought in, put z for the number of hogſheads of the worſe ſort; and ſince the whole number was $11\frac{1}{2}$, the number of hogſheads of the better ſort was $11\frac{1}{2}-z$: but if one hogſhead of the worſe ſort coſts 5 pounds, z will coſt $5z$ in pounds; in like manner, $11\frac{1}{2}-z$, the quantity of wine of the better ſort, will coſt $92-8z$ in pounds; therefore the whole ſum laid out was $5z+92-8z$, that is, $92-3z$ in pounds; but we found above that the ſum laid out was $72\frac{1}{4}$ pounds; therefore $72\frac{1}{4}=92-3z$; therefore $3z=19\frac{3}{4}$, and $z=6\frac{7}{12}$: therefore if we allow 60 gallons to the hogſhead, the quantity of wine bought in of the worſe ſort was 6 hogſheads 35 gallons; ſubtract this from the whole quantity, 11 hogſheads 30 gallons, and there remains 4 hogſheads 55 gallons of the better ſort, or $4\frac{11}{12}$.

For

For a further proof of this, multiply $6\frac{7}{12}$, the quantity of the worſe ſort, by 5, and it's price will be $32\frac{11}{12}$ pounds, that is, 32*l*. 18*s*. 04*d*: multiply alſo $4\frac{11}{12}$, the quantity of the better ſort, by 8, and it's price will be $39\frac{4}{12}$ or $39\frac{1}{3}$ pounds, that is, 39*l*. 06*s*. 08*d*; add theſe two prices together, *viz*. 32*l*. 18*s*. 04*d* and 39*l*. 06*s*. 08*d*, and the amount will be 72*l*. 05*s*. 00*d*, as it ought.

SCHOLIUM.

For a further explication of the foregoing ſolution, we had there two limitations of the value of y, to wit, that $yy-22y+121$ muſt be greater than 61, and that $yy-24y+144$ muſt be leſs than 84. Now if $yy-22y+121$ be greater than 61, then according to art. 232, $y-11$ muſt either be greater than $+\sqrt{61}$ or leſs than $-\sqrt{61}$; the former caſe was conſidered in the foregoing ſolution, but not the latter, to wit, that $y-11$ muſt be leſs than $-\sqrt{61}$: let us make $y-11$ leſs than $-7\frac{9}{10}$, and it will of courſe be leſs than $-\sqrt{61}$, becauſe $7\frac{9}{10}$ is ſomewhat greater than $\sqrt{61}$: now if $y-11$ be leſs than $-7\frac{9}{10}$, y muſt be leſs than $3\frac{1}{10}$. Again, if $yy-24y+144$ be leſs than 84, then $y-12$ muſt be leſs than $+\sqrt{84}$, and greater than $-\sqrt{84}$; the latter caſe is here to be conſidered: let us then make $y-12$ greater than $-9\frac{1}{10}$, ſince $9\frac{1}{10}$ is ſomewhat leſs than $\sqrt{84}$, and we ſhall have y greater than $2\frac{9}{10}$ or $3-\frac{1}{10}$. Here then we have another value of y different from that which was made uſe of in the foregoing ſolution; for here we learn, that y may be made equal to any number greater than $3-\frac{1}{10}$, and leſs than $3+\frac{1}{10}$: let us then make $y=3$, that is, let $xx-60$ be equated to a ſquare whoſe ſide is $x-3$, and we ſhall have $x=11\frac{1}{2}$ as before.

Had we allowed the quantity x all the liberty it had a right to, *viz*. of being any number greater than $10\frac{2}{10}$ and leſs than $12\frac{2}{10}$, the limits of

of y derived from this consideration would also have been enlarged from $2\frac{7}{10}$ to $3\frac{3}{10}$; therefore the less value of y may be taken equal to any number between $2\frac{7}{10}$ and $3\frac{3}{10}$, and the greater equal to any number between $18\frac{1}{10}$ and $22\frac{7}{10}$. If y be taken out of these limits, the consequence will be, that x will be driven out of it's limits, and the quantity of wine of one sort or the other will be found negative, which is absurd.

PROBLEM 30.

Being here proposed only as it introduces a sort of duplicate equality to be resolved, not hitherto taken notice of.

260. *To find a number which being severally multiplied by eight and eighteen, and nine being added to the former product and subtracted from the latter, both the sum and the remainder shall be square numbers.*

SOLUTION.

Put x for the number sought, and we shall have this duplicate equality, $8x+9=\square$, and $18x-9=\square$, which is different from any of the forms hitherto described, and is only resolvable upon this account, to wit, that 8 and 18, the two coefficients of x, are to each other as two square numbers, that is, as 4 to 9: for since 8 is to 18 as 4 to 9, 8×9, the product of the extremes, will be equal to 18×4, the product of the two middle terms; and therefore if the first square $8x+9$ be multiplied into 9, and the second square $18x-9$ be multiplied into 4, the products $72x+81$ and $72x-36$ will still be square numbers, the multiplicants 9 and 4 being squares; and the coefficients of x in both numbers will now be the same, to wit 72; and the duplicate equality in the former case will now be reduced to this, *viz.* to make $72x+81=\square$, and $72x-36=\square$. Now the difference between these two quantities is 117; and therefore if we find two squares whose difference is 117, and equate $72x-36$ to the lesser of those two squares, $72x+81$ must be equal to the greater, and so we shall have both $72x-36=\square$, and $72x+81=\square$. Now to find two square numbers whose difference is 117, I proceed according to art. 231, and resolve 117 into the two factors 39 and 3, whose difference is 36, and half difference 18; and as the square of 18 is 324, I have this equation, $72x-36=324$; whence $x=5$, which is such a number as the problem requires: for if 5 be severally multiplied by 8 and 18, the products will be 40 and 90; and if 9 be added to the former product and subtracted from the latter, the sum will be 49 and the remainder 81, both square numbers.

THE

THE

ELEMENTS of ALGEBRA

BOOK VII.

Of PROPORTION.

Of the neceſſity of reſuming the doctrine of proportion, and removing ſome difficulties which ſeem to attend it as delivered in the Elements.

261. IN the 15th and 16th articles of this treatiſe I have laid down as clearly, and yet as ſuccinctly as I was able, the doctrine of proportion ſo far as it relates to numbers and commenſurable quantities, whereof any one may be conſidered as ſome multiple, part or parts of another of the ſame kind; and it ſerved well enough all the purpoſes it was deſigned for. But being in the next book to apply Algebra to Geometry, and ſo to conſider proportion as it relates to magnitudes in general whether commenſurable or incommenſurable, I ſhould come ſhort of the ἀκρίβεια *geometrica*; was I not to reſume this ſubject, and to conſider it now in it's full extent as it is laid down in the fifth book of the elements of Geometry. I might indeed have excuſed myſelf from this part of my taſk, and ſhould have been very glad to have done it, by referring the reader at once to the elements themſelves without any further aſſiſtance; but I could not withſtand ſome reaſons drawn from experience, which to me ſeemed to plead very powerfully to the contrary.

I frequently obſerve that moſt of thoſe who ſet themſelves to read *Euclid*, when they come at the fifth book which treats of proportion, either entirely paſs it by as containing ſomething too ſubtil to be comprehended by young beginners, or elſe touch ſo very ſlightly upon it as to be little the better for it; and thus the doctrine of proportion (which is certainly the moſt extenſive, and conſequently the moſt uſeful part of the Mathematics) is either taken for granted, or at beſt but partially underſtood by them.

them. These ſchemes there made uſe of are ſcarce bold enough, I had almoſt ſaid, ſcarce complicated enough to affect the imagination ſo ſtrongly as is neceſſary to fix the attention.

The firſt, ſecond, third, fifth and ſixth propoſitions are ſelf-evident, as well as ſome others, and upon that very account create an impatient reader much greater uneaſineſs than if they were further removed from common ſenſe; becauſe the truths from whence theſe propoſitions are deduced are not ſo diſtinct from the propoſitions themſelves as in many other caſes. But it ought to be conſidered, that the perfection of all arts and ſciences in general, and of Geometry in particular, is to ſubſiſt upon as few firſt principles or axioms as is poſſible; and therefore whenever a propoſition, how evident ſoever it may appear in itſelf, can be deduced from any that is gone before, it ought by all means to be ſo deduced, and not to be made a firſt principle, and ſo unneceſſarily to increaſe their number.

The deſign of a geometrical demonſtration is not ſo much to illuſtrate the propoſition to which it is annexed, or to render it more evident than it would have been without it, (though this ought certainly to be done wherever the nature of things will permit,) as it is to ſhew the neceſſary connexion the propoſition to be demonſtrated has with ſome previous truth already admitted or proved, ſo as to ſtand and fall together, whether ſuch previous truths be more or leſs evident than the propoſition to be demonſtrated: I ſay more or leſs evident; for it is not uncommon in the courſe of *Euclid*'s geometry to meet with propoſitions demonſtrated from others that are leſs evident than themſelves. For an inſtance of this we need go no further than the twentieth propoſition of the firſt book, where it is demonſtrated that *in every triangle any two ſides taken together are greater than the third*: now it is certain that this propoſition is more evident than that the external angle is greater than either of the internal and oppoſite ones; and yet the former, by the help of the 19th propoſition, is demonſtrated from the latter.

But there is another reaſon to be given for demonſtrating ſelf-evident propoſitions in many caſes, and particularly in this fifth book of the elements. A propoſition may ſometimes be taken to be ſelf-evident according to our narrow and ſcanty notions of things, which when better underſtood, will be found to be otherwiſe. Theſe propoſitions, to wit, that *equal quantities will have the ſame proportion to a third*, that *of two unequal quantities the greater will have a greater proportion to a third than the leſs*, and ſome others of the ſame ſtamp in the fifth book, are ſuch as will paſs with moſt for ſelf-evident propoſitions; and ſo they are without all doubt according to the common conception of proportionality; but when they come to be examined according to the juſteſt and more extenſive

tensive idea *Euclid* has given of it, I fear they will both, and the latter more especially, be found to want demonstration.

In a perfect and regular system of elementary Geometry, such a one as that of *Euclid* may be supposed to be, or at least to have been, certain properties of lines, angles and figures are to be laid down, and those of the simplest kind, for definitions; from whence, and from one another, all the rest are to be derived with the utmost rigour, without the least appeal even to common sense. Common sense is by no means to be made the standard of any geometrical truths whatever, except first principles: it's province must be only to judge whether a proposition be duly demonstrated according to the rules already prescribed, that is, whether the necessary connexion it has with any previous truth be clearly and distinctly made out; when that is done, nothing remains but to pass sentence. Whilst the science continues thus circumscribed, no mistakes, no disputes can arise concerning it's boundaries; but whenever these come to be transgressed, such a loose will be given to Geometry that it would be impossible to agree upon any others whereby to restrain it.

Thus much I thought proper to lay down concerning the nature of a geometrical demonstration, that young students may not sometimes think themselves disappointed, or not proceed with that coolness and judgement absolutely necessary to conduct them through the elements of Geometry.

But as to the matter in hand, there is another difficulty still behind, which I believe, is often a greater discouragement to young beginners in their entrance into the doctrine of proportion, than any which have hitherto been alledged, and that is the difficulty of understanding and applying *Euclid*'s definition of proportionable quantities. But to take away all excuse from this quarter, I have here annexed a small dissertation conducing (as I take it) to clear up that definition. It is an extract out of some loose papers I have by me; and therefore the reader must not be surprized if he finds some things repeated here which have already been mentioned in another part of this book.

A vindication of the fifth definition of the fifth book of Euclid's *elements.*

262. *N. B.* For a more distinct understanding of what follows, it must be observed, that *By a part, in the sense of the fifth book of* Euclid, *is meant an aliquot part, and not a part as part related to some whole.* Thus 3 is a part of 12 in *Euclid*'s sense, as being just four times contained in it; and though 9 be a part of 12 in the same sense as the part is distinguished from the whole, yet 9 in *Euclid*'s sense is not a part, but parts of 12, as being three fourth parts of it.

1st. *If two quantities* A *and* B *be commensurable, then* A *must necessarily be either some multiple, or some part, or some parts of* B. For if *A* and *B* be commensurable, then either *B* must measure *A*, or *A* must measure *B*, or they must both be measured by some third quantity: if *B* measures *A* any number of times, suppose 3 times, then *A* will be equal to 3 times *B*, and consequently will be a multiple of *B*: if *A* measures *B* any number of times, suppose 3 times, then *A* will be a third part of *B*, and consequently will be a part of *B*: if *A* and *B* do not measure one the other, let *C* measure them both, and let *C* be contained exactly in *A* 3 times and in *B* 4 times; then will a third part of *A* be equal to a fourth part of *B*, as being both equal to *C*; multiply both sides of the equation by 3, and you will have $\frac{3}{3}$ of *A* or *A* equal to $\frac{3}{4}$ of *B*; therefore in this case *A* is said to be parts of *B*.

2dly. *If two quantities* A *and* B *are incommensurable, then* A *can neither be any multiple of* B, *nor any part or parts of it.* For if *A* was any multiple of *B*, then *B* would measure both itself and *A*, which contradicts the supposition of their incommensurability: in like manner, if *A* was any part of *B*, then *A* would measure both itself and *B*: in the last place I say that neither can *A* be any parts of *B*; for if *A* was any parts of *B*, suppose $\frac{3}{4}$ of *B*, then $\frac{1}{4}$ of *B* would measure both *A* and *B*, which still contradicts the supposition: *A* indeed may be greater or less than some part or parts of *B*, but can never be equal to any; so subtil is the composition of continued quantity. As for instance; it is demonstrated in art. 200, that the side and diagonal line of a square are incommensurable to each other: let then *A* be the diagonal of a square whose side is *B*, and the square of *A* will be to the square of *B* as 2 to 1, as is evident from the 47th of the first book of *Euclid*; therefore *A* will be to *B* as the square root of 2 is to 1; but the square root of 2 is 1.414 &c, that is $\frac{14}{10}$, or more nearly $\frac{141}{100}$, or more nearly still $\frac{1414}{1000}$: whence it follows, that if the side of a square be divided into 10 equal parts, the diagonal will contain more than 14 of these parts, but not so much as 15 of them; if the side be divided into 100 equal parts, the diagonal will contain above 141 of such parts, but not 142; if the side be divided into 1000 equal parts, the diagonal will contain above 1414 of such parts, but not 1415; and so on *ad infinitum:* therefore the diagonal of a square can never be exactly expressed by parts of the side any more than the side can by parts of the diagonal. The side may indeed be set off upon the diagonal, and so be considered as part of it, so far as part of the whole; but the side can never be exactly expressed by any number of aliquot parts of the diagonal, be these parts ever so small. Limits may be found and expressed by parts of the diagonal as near as possible to each

each other, between which the ſide ſhall always conſiſt, and by which it may be expreſſed to any degree of exactneſs except perfect exactneſs. And thus alſo may approximations be made in the expreſſions of many other incommenſurable quantities one by another.

3dly. From the laſt ſection it appears, that *If two quantities* A *and* B *be incommenſurable, no multiple of one can ever be equal to any multiple of the other.* For if, for inſtance, $4A$ could be equal to $3B$, then (dividing by 4) A would be found to be juſt $\frac{3}{4}$ of B, contrary to what has been above demonſtrated.

4thly. *If four quantities* A, B, C *and* D *be ſuch, that* A *is the ſame part or parts of* B *that* C *is of* D, *then are thoſe four quantities* A, B, C *and* D *ſaid to be proportionable, or* A *is ſaid to have the ſame proportion to* B *that* C *hath to* D. Thus if A be a fourth part of B, and C a fourth part of D, then A will be the ſame part of B that C is of D, and they will be proportionable. Thus again, if $A = \frac{3}{4}B$, and $C = \frac{3}{4}D$, or if $A = \frac{8}{4}B$ or $2B$, and $C = \frac{8}{4}D$ or $2D$, or if $A = \frac{11}{4}B$, and $C = \frac{11}{4}D$, in all theſe inſtances (comprehending multiples under the notion of parts) A may be ſaid to be the ſame parts of B that C is of D; and therefore according to this definition, A hath the ſame proportion to B that C hath to D; which is true, and the mark of proportionality here given is infallible, but not adequate to our idea of it; for though this mark be never found without proportionality, yet proportionality is often found without this mark. Proportionality is often found among incommenſurables; but it can never be tried or proved by the marks here given. I believe no body ever doubted that the ſide of one ſquare hath the ſame proportion to it's diagonal that the ſide of any other ſquare hath to it's diagonal; and therefore A may have the ſame proportion to B that C hath to D, though A be incommenſurable to B and C to D: yet who can ſay in this caſe, that A is the ſame part or parts of B that C is of D, when it has already been ſhewn, that A is no part or parts of B, nor C of D? This way therefore of defining proportionable quantities by a ſimilitude of aliquot parts, cannot (in ſtrictneſs of Geometry) be laid down as a proper foundation, ſo as from thence to derive all the other properties of proportionality: for ſince theſe properties are to be applied to incommenſurable as well as commenſurable quantities, it is fit they ſhould be deduced from a fundamental property that relates equally to both.

5thly. In order then to eſtabliſh a more general character of proportionality, I ſhall aſſume the following principle which equally relates to commenſurable and incommenſurable quantities; and which, I believe, there

 is

is no one who has a juſt idea of proportionality, which way ſoever he may chuſe to expreſs it, or whether he can expreſs it or not, but will eaſily allow me, which is; that *If four quantities* A, B, C *and* D *be proportionable, that is, if* A *has the ſame proportion to* B *that* C *hath to* D, *it will then be impoſſible for* A *to be greater than any part or parts of* B, *but* C *muſt alſo be greater than a like part or parts of* D; *or for* A *to be equal to any part or parts of* B, *but that* C *muſt alſo be equal to a like part or parts of* D; *or for* A *to be leſs than any part or parts of* B, *but that* C *muſt alſo be leſs than a like part or parts of* D. Thus if *A* hath the ſame proportion to *B* that *C* hath to *D*, it will then be impoſſible for *A* to be greater than, equal to, or leſs than $\frac{14}{10}$ of *B*, but *C* muſt alſo be greater than, equal to, or leſs than $\frac{14}{10}$ of *D*. This principle, I ſay, is ſo very clear that nothing more needs to be ſaid of it, either by way of explication or demonſtration: and if by the help hereof I can demonſtrate the converſe, we ſhall then have a general mark of proportionality as extenſive as proportionality itſelf. Now the converſe of the foregoing propoſition is this; *If there be four quantities* A, B, C *and* D, *and if the nature of theſe quantities be ſuch, that* A *cannot poſſibly be greater than, equal to, or leſs than any part or parts of* B, *but at the ſame time* C *muſt neceſſarily be greater than, equal to, or leſs than a like part or parts of* D, *let the number or denomination of theſe parts be what they will; I ſay then, that* A *muſt neceſſarily have the ſame proportion to* B *that* C *hath to* D. If this be denied, let ſome other quantity *E* have the ſame proportion to *D* that *A* hath to *B*, that is, let *A*, *B*, *E* and *D* be proportionable quantities; then imagining the quantity *D* to be divided into any number of equal parts, ſuppoſe 10, let *E* be greater than 14 of theſe parts and leſs than 15, that is, let *E* be greater than $\frac{14}{10}$ and leſs than $\frac{15}{10}$ of *D*; then muſt *A* neceſſarily be greater than $\frac{14}{10}$ and leſs than $\frac{15}{10}$ of *B*: this is evident from the conceſſion already made, ſince *A* is ſuppoſed to have the ſame proportion to *B* that *E* hath to *D*. But if *A* be greater than $\frac{14}{10}$ and leſs than $\frac{15}{10}$ of *B*, then *C* muſt be greater than $\frac{14}{10}$ and leſs than $\frac{15}{10}$ of *D* by the *hypotheſis*; the relation between *A*, *B*, *C* and *D* being ſuppoſed to be ſuch, that *A* cannot be greater or leſs than any part or parts of *B*, but *C* accordingly muſt be greater or leſs than a like part or parts of

D.

D. Therefore we are now advanced thus far, that if *E* lies between $\frac{14}{10}$ and $\frac{15}{10}$ of *D*, *C* muſt alſo neceſſarily lie betwixt the ſame limits: now the difference betwixt $\frac{14}{10}$ and $\frac{15}{10}$ of *D* is $\frac{1}{10}$ of *D*; therefore the difference betwixt *C* and *E*, which lie both between theſe two limits muſt be leſs than $\frac{1}{10}$ of *D*. This is upon a ſuppoſition that the quantity *D* was at firſt divided into 10 equal parts; but if inſtead of 10 we had ſuppoſed it to have been divided into 100, or 1000, or 10000 equal parts, (which ſuppoſitions could not have affected the quantities *C* and *E*,) the concluſion would then have been, that the difference betwixt *C* and *E* would have been leſs than the hundredth, or thouſandth, or ten thouſandth part of *D*; and ſo on *ad infinitum*: therefore the difference between *C* and *E* (if there be any difference) muſt be leſs than any part of *D* whatever; therefore the difference between *C* and *E* is only imaginary, and not real; therefore in reality *C* is equal to *E*. Since then *C* is equal to *E*, and that *A* is to *B* as *E* is to *D*, the conſequence muſt be that *A* is to *B* as *C* is to *D*. *Q. E. D.*

Here then we have a proper characteriſtic of proportionality which always accompanies it, and on the other hand, is never to be found without it, to wit, that four quantities may be ſaid to be proportionable, the the firſt to the ſecond as the third is to the fourth, when the firſt cannot be greater than, equal to, or leſs than any part or parts of the ſecond, but the third muſt accordingly be greater than, equal to, or leſs than a like part or parts of the fourth: or thus; *Four quantities may be ſaid to be proportionable as above, when the firſt cannot be contained between two limits expreſſed by any parts of the ſecond, how near ſoever theſe limits may approach to each other, but the third muſt neceſſarily be contained between the limits expreſſed by like parts of the fourth.*

6thly. Had *Euclid* ſtopped here, without refining any further upon the criterion of proportionality delivered in the laſt ſection, (for I dare venture to affirm he was no ſtranger to it,) I doubt not but it would have given much greater ſatisfaction to the generality of his diſciples, eſpecially thoſe of a leſs delicate taſte, than that which he advances in the fifth book of his elements, as being more cloſely connected with the common idea of proportionality: but it was eaſy to ſee, that in demonſtrating ſeveral other affections of proportionable quantities upon this ſcheme, there would then be frequent occaſion for taking ſuch and ſuch parts of magnitudes, as there is now for taking ſuch and ſuch multiples of them,

 the

the *praxis* of which partition had no where as yet been taught by *Euclid*; nay he rather ſeems to have determined, as far as poſſible, to avoid it, and that upon no ill grounds neither; for the uſe of whole numbers is in all caſes juſtly eſteemed more natural and more elegant than that of fractions, and the multiplication of quantities has always been looked upon as more ſimple in the conception than the reſolution of them into their aliquot parts. It is for this reaſon that *Euclid* never ſhews how to multiply a line or any other quantity whatever, aſſuming the *praxis* thereof as a ſort of *poſtulatum*; whereas in the ninth propoſition of the ſixth book of his elements he ſhews how to cut off any aliquot part of any given line whatever. Upon theſe and ſuch like conſiderations it was that *Euclid* reſolved to advance his characteriſtic property of proportionality one ſtep higher, by ſubſtituting multiples inſtead of aliquot parts in ſuch a manner as we ſhall now deſcribe; and we ſhall at the ſame time demonſtrate the juſtneſs of his definition from what has been already laid down in the laſt ſection. The propoſition to be demonſtrated ſhall be this: *If there be four quantities* A, B, C *and* D, *whereof* EA *and* EC *are any equimultiples of the firſt and third, and* FB *and* FD *are any other equimultiples of the ſecond and fourth; and if now theſe quantities are of ſuch a nature, that* EA *cannot be greater than, equal to, or leſs than* FB, *but at the ſame time* EC *muſt neceſſarily be greater than, equal to, or leſs than* FD, *when compared reſpectively, be the multiplicators* E *and* F *what they will: I ſay then that* A *muſt neceſſarily have the ſame proportion to* B *that* C *hath to* D. Now that four quantities may be under ſuch circumſtances as are here deſcribed, can be queſtioned by no one who has with any attention conſidered the nature of proportionable quantities: for ſuppoſe A to be the diameter and B the circumference of any circle, and C to be the diameter and D the circumference of any other circle; who doubts but that twentytwo times the diameter of one circle will be greater than, equal to, or leſs than ſeven times the circumference, according as twentytwo times the diameter of the other circle is greater than, equal to, or leſs than ſeven times the circumference of that circle? I now proceed to the demonſtration of the propoſition.

If it be denied that A is to B as C is to D, let A be to B as G is to D; and then ſuppoſing D to be divided into 10 equal parts, let G be greater than 14 of theſe parts and leſs than 15: then ſince by the ſuppoſition A is to B as G is to D, we ſhall have A greater than $\frac{14}{10}$ and leſs than $\frac{15}{10}$ of B; therefore $10A$ will be greater than $14B$ and leſs than $15B$; but by the *hypotheſis*, no multiple of A can be greater or leſs than any

any multiple of B, but the ſame multiple of C muſt be greater or leſs than a like multiple of D; therefore $10C$ is greater than $14D$ and leſs than $15D$; therefore C is greater than $\frac{14}{10}$ and leſs than $\frac{15}{10}$ of D; therefore if G be a quantity between $\frac{14}{10}$ and $\frac{15}{10}$ of D, C muſt alſo be a quantity between the ſame limits; therefore the difference betwixt C and G muſt be leſs than $\frac{1}{10}$ of D. This is upon a ſuppoſition that D was divided into 10 equal parts; but C and G will be the ſame, into what number of parts ſoever we ſuppoſe D to be divided; therefore if we ſuppoſe D to be divided into 100, 1000, or 10000 equal parts, *&c*, the difference betwixt C and G might have been ſhewn to be leſs than the hundredth, or the thouſandth, or the tenthouſandth part of D; and ſo on *ad infinitum*; therefore C and G are equal, as was ſhewn in the 5th ſection. Since then A cannot be greater than, equal to, or leſs than any part or parts of B, but G muſt be greater than, equal to, or leſs than a like part or parts of D, becauſe A is to B as G is to D; and ſince G cannot be greater than, equal to, or leſs than any part or parts of D, but C muſt be greater than, equal to, or leſs than the ſame part or parts of D, becauſe G and C are equal; it follows *ex æquo*, that A cannot be greater than, equal to, or leſs than any part or parts of B, but that C muſt accordingly be greater than, equal to, or leſs than a like part or parts of D; and conſequently that A is to B as C is to D, according to the mark of proportionality given in the laſt ſection. *Q. E. D.*

Four quantities then may be ſaid to be proportionable, the firſt to the ſecond as the third to the fourth, when no equimultiples whatever can be taken of the firſt and third, but what muſt either be both greater than, or both equal to, or both leſs than any other equimultiples that can poſſibly be taken of the ſecond and fourth, when compared reſpectively.

7thly. As number is a diſcrete, and not a continued quantity, there is ſuch a thing as a *minimum* in the parts of number, whereas in thoſe of extenſion there is none; whence it follows, that the parts of number muſt neceſſarily be more diſtinct, and for that reaſon more aſſignable than are the parts of extenſion. Again, as all numbers are commenſurable by unity, every number may be conceived either as ſome multiple, or ſome part, or ſome parts of every other. Hence it is that *Euclid*, defining proportionable numbers, makes uſe of the definition given in the 4th ſection; ſo unwilling was he to recede from the common notion of proportionable quantities, whenever the ſubject he treated of would bear it.

Of

Of the seventh definition of the fifth book of Euclid.

263. If it be allowed to be a sufficient mark of the proportionality of four quantities, when they are so related to one another in their own natures, that no equimultiples can be taken of the first and third, but what must either be both greater than, or both equal to, or both less than any other equimultiples that can possibly be taken of the second and fourth; then wherever it happens, or may happen otherwise, there can be no proportionality. As for instance, *If in comparing equimultiples of the first and third with other equimultiples of the second and fourth, there be any cases wherein the first multiple shall be greater than the second, and yet the third not greater than the fourth; or wherein the first multiple shall be less than the second, and the third not less than the fourth; then the first quantity will not have the same proportion to the second that the third hath to the fourth, but either a greater as in the former case, or a less as in the latter.* Nay, and I may add further, that *If of four quantities, the first hath a greater proportion to the second than the third hath to the fourth, there must be cases existing, whether those cases can be assigned or not, wherein of equimultiples of the first and third, and of other equimultiples of the second and fourth, the first multiple shall exceed the second, and yet the third shall not exceed the fourth:* for if no such cases were possible, then the first quantity must either have the same proportion to the second that the third hath to the fourth, or a less; both which are contrary to the supposition. Thus we have found the fifth and seventh definitions of the fifth book of the elements both of a piece.

A question arising out of the foregoing article.

264. This is all that was necessary to be observed concerning the foregoing definitions: but if, having given four quantities A, B, C and D, whereof A hath a greater proportion to B than C hath to D, any one for his own private satisfaction would know how to find such equimultiples of A and C, and such other equimultiples of B and D, that A's multiple shall exceed that of B, and at the same time C's multiple shall not exceed that of D, it may be done thus. If the quantities A, B, C and D be commensurable, let their ratios be expressed by numbers: as for instance, let A be to B as 7 to 5, and let C be to D as 4 to 3; then will 4 and 3, the numeral expressions of the lesser ratio, be the multiplicators required, if of the terms A and B, the greater term A be multiplied into the lesser multiplicator 3, and the lesser term B into the greater multiplicator 4; for then $3A$ (21) will be greater than $4B$ (20), and yet $3C$ (12) will not be greater than $4D$ (12), for the

two

two laſt multiples are equal. But if ſuch multiples be required, that the firſt multiple ſhall be greater than the ſecond, and at the ſame time the third multiple ſhall be leſs than the fourth, then ſome intermediate fraction muſt be taken between $\frac{7}{5}$ and $\frac{4}{3}$, and the terms of ſuch a fraction will be the multiplicators required. As for inſtance, throwing the extreme fractions into decimals, we have $\frac{7}{5} = 1.4$, and $\frac{4}{3} = 1.34$ —; therefore if any decimal fraction be taken between 1.4 and 1.34, ſuch a fraction being reduced to integral terms will give the multiplicators required. Let us aſſume 1.375, that is, $\frac{1375}{1000}$ or $\frac{11}{8}$; then will $8A\,(56)$ be greater than $11B\,(55)$, and at the ſame time $8C\,(32)$ will be leſs than $11D\,(33)$.

If the quantity A be incommenſurable to B, or C to D, or both to both, find however, by ſcholium the ſecond in art. 179, ſuch numbers as will expreſs theſe ratios as accurately as occaſion requires. As let the ratio of the number E to the number F be nearly the ſame with that of A to B; and let the ratio of the number G to the number H be nearly the ſame with that of C to D: then if either of theſe ratios, to wit, the ratio of E to F, or the ratio of G to H, lie between the ratios of A to B and of C to D, the terms of the intermediate ratio will make proper multiplicators; but if neither of theſe caſes happen, ſome intermediate fraction muſt be taken between the two fractions $\frac{E}{F}$ and $\frac{G}{H}$.

Having thus prepared my young ſtudent for *Euclid's* doctrine of proportion, partly by ſetting him right in his notions of things, and partly by removing out of his way all that rubbiſh which ſeemed to block up his entrance to it, I hope I ſhall now be able to conduct him through the whole with a great deal of eaſe, and that he will meet with fewer difficulties in reading the following propoſitions than an equal number in any other part of the elements: and yet all I have done herein has been only to mitigate, as far as I thought proper, the rigour and ſeverity of the author's manner of writing, and to render his demonſtrations more eaſy to the imagination, which the compiler in his whole ſyſtem ſeems to have had no great tenderneſs for: but whatever I have done elſe, I have taken care to preſerve the force of the demonſtrations, and I hope, in a great meaſure, their elegancy too. I have uſed no algebraic computations in demonſtrating theſe propoſitions, except what may be juſtified by the antecedent ones; as well knowing that theſe principles were never intended to depend upon arithmetical operations, but rather arithmetical operations upon them. I have however, for the reader's eaſe, made uſe of the ſimpleſt algebraic notation. Thus A, B, C, D ſignify magnitudes

of any kind whatever; *E*, *F*, *G*, *H* always ſignify whole numbers, unleſs where notice is given to the contrary; $A+B$ ſignifies the ſum of any two homogeneous magnitudes *A* and *B*; $A-B$ their difference, or the exceſs of *A* above *B*; *EA* and *FB* ſignify any two multiples of *A* and *B*, the multiplicators being *E* and *F*; *&c.* I have ſometimes alſo uſed very eaſy conſequences of this notation; as that if $A-B$ be added to *B*, the ſum will be *A*, which indeed is a general axiom, and ſaying no more than that if to any magnitude be added the exceſs of a greater above it, the ſum will be the greater magnitude.

THE FIFTH BOOK OF *EUCLID*'s ELEMENTS.

DEFINITIONS.

265. 1. *A leſſer magnitude is ſaid to be a part of a greater, when the leſſer meaſures the greater.*

2. *A greater magnitude is ſaid to be a multiple of a leſs, when the greater is meaſured by the leſs.*

Note. Our language is not nice enough to expreſs theſe two definitions as they are in the Greek and Latin.

We may further obſerve, that by theſe two definitions every ſimple quantity is excluded from being conſidered either as a part or a multiple of itſelf; for to be a part, in this ſenſe, is to be leſs than that whereof it is a part, and to be a multiple is to be greater than that whereof it is a multiple.

3. *Ratio is that mutual relation two homogeneous quantities are in, when compared together in reſpect to their quantity.* Thus the exceſs of 2 above 1 is equal to the exceſs of 4 above 3, and yet the ratio of 2 to 1 is greater than the ratio of 4 to 3; that is, 2 has more magnitude when compared with 1 than 4 hath when compared with 3; ſince 2 is double of 1, and 4 is not double of 3. But on the other hand, 3 hath a greater ratio to 4 than 1 hath to 2, becauſe 3 hath more magnitude in compariſon of 4 than 1 hath in compariſon of 2; for 3 is more than the half of 4, whereas 1 is but juſt the half of 2.

4. *All quantities are ſaid to be in ſome ratio or other, when they are capable of being ſo multiplied as to exceed one another.*

Note. By this definition, 1ſt, All heterogeneous quantities are excluded from having any ratio one to another, becauſe heterogeneous quantities are ſuch, that their multiples are no more capable of compariſon as to exceſs and defect, than the quantities themſelves: a yard can never be multiplied till it exceeds an hour, *&c.* 2dly, All infinitely ſmall quantities are hereby excluded from having any ratio to finite ones, becauſe the former can never be ſo multiplied as to exceed the latter.

5. *Mag-*

5. *Magnitudes are ſaid to be in the ſame ratio, the firſt to the ſecond as the third to the fourth, when no equimultiples can be taken of the firſt and third; but what muſt either be both greater than, or both equal to, or both leſs than any other equimultiples that can poſſibly be taken of the ſecond and fourth.*

Note. This and the ſeventh definition have been explained already.

6. *Magnitudes in the ſame ratio may be called proportionals.*

7. *If there be four quantities, whereof equimultiples are taken of the firſt and third, and other equimultiples of the ſecond and fourth; and if any caſe can be aſſigned, wherein the multiple of the firſt ſhall be greater than the multiple of the ſecond, and at the ſame time the multiple of the third ſhall not be greater than the multiple of the fourth; then of theſe four quantities, the firſt is ſaid to have a greater ratio to the ſecond than the third hath to the fourth.*

8. *Proportion conſiſts in a ſimilitude of ratios.*

9. *Proportion cannot be expreſſed in fewer than three terms:* as when we ſay that A is to B as B is to C.

10. *Whenever three quantities are continual proportionals, the firſt is ſaid to be to the third in a duplicate ratio of the firſt to the ſecond: and on the other hand, the firſt is ſaid to be to the ſecond in a ſubduplicate ratio of the firſt to the third.*

11. *If four quantities be continual proportionals, the firſt is ſaid to be to the fourth in a triplicate ratio of the firſt to the ſecond; and ſo on.*

12. *The antecedents of all proportions are called homologous terms; and ſo alſo are the conſequents: but antecedents and conſequents conſidered together, are never called homologous terms, but heterologous.*

Note. Theſe three laſt definitions, though placed here, have nothing to do in the following fifth book, but in the ſixth.

13. *Alternate proportion is, when four quantities being proportionable, the firſt to the ſecond as the third to the fourth, it is concluded, that the firſt is to the third as the ſecond to the fourth;* the juſtneſs of which concluſion, as well as of all the reſt that follow, will be ſufficiently made out in the following propoſitions.

14. *Inverſe proportion is, when four quantities being proportionable, the firſt to the ſecond as the third to the fourth, it is concluded, that the ſecond is to the firſt as the fourth to the third.*

15. *Compoſition of proportion is, when four quantities being proportionable, the firſt to the ſecond as the third to the fourth, it is concluded, that the ſum of the firſt and ſecond is to the ſecond as the ſum of the third and fourth is to the fourth.*

16. *Diviſion of proportion is, when four quantities being proportionable, the firſt to the ſecond as the third to the fourth, it is concluded, that the ex-*

cess of the first above the second is to the second as the excess of the third above the fourth is to the fourth.

17. *Conversion of proportion is, when four quantities being proportionable, the first to the second as the third to the fourth, it is concluded, that the first is to the excess of the first above the second as the third is to the excess of the third above the fourth.*

18. *If ever so many quantities in one series be compared with as many in another; and if from all the ratios in one being equal to all those in the other, either in the same or a different order, it be concluded, that the extremes in one series are in the same proportion with the extremes in the other, this proportionality of the extremes is said to follow* ex æquo, *or* ex æqualitate rationum.

19. *If all the ratios in one series be equal to all those in the other, and in the same order, this is called ordinate proportion; and the extremes in this case are said to be proportionable* ex æquo ordinate, *or barely* ex æquo.

20. *If all the ratios in one series be equal to all those in the other, but not in the same order, this is called inordinate proportion; and the extremes are said to be proportionable* ex æquo perturbate.

Thus if *A*, *B* and *C* in one series be compared with *D*, *E* and *F* in another; and if *A* is to *B* as *D* to *E*, and *B* to *C* as *E* to *F*, this is called ordinate proportion, and *A* is said to be to *C* as *D* to *F ex æquo ordinate*, or barely *ex æquo:* but if *A* is to *B* as *E* to *F*, and *B* to *C* as *D* to *E*, this is called inordinate proportion, and *A* is said to be to *C* as *D* to *F ex æquo perturbate.*

PROPOSITION I.

266. *If there be ever so many homogeneous quantities* A, B, C, *whereof* EA, EB, EC *are equimultiples respectively; I say then, that the sum* $\overline{EA+EB+EC}$ *will be the same multiple of the sum* $\overline{A+B+C}$ *that* EA *is of* A, *or* EB *of* B, &c.

For the multiples *EA*, *EB* and *EC* may be considered as so many distinct heaps or parcels, whereof *EA* consists wholly of *A*s, *EB* of *B*s, and *EC* of *C*s; and since the number of *A*s in *EA* is the same with the number of *B*s in *EB*, or of *C*s in *EC*, it follows, that as often as *A* can be singly taken out of *EA*, or *B* out of *EB*, or *C* out of *EC*, just so often may the whole sum $\overline{A+B+C}$ be taken out of the whole sum $\overline{EA+EB+EC}$; therefore the sum $\overline{EA+EB+EC}$ is the same multiple of the sum $\overline{A+B+C}$ that *EA* is of *A*, or *EB* of *B*, &c. *Q. E. D.*

PRO-

PROPOSITION 2.

267. *If* EA *and* EB *be equimultiples of any two quantities whatever* A *and* B, *and if* FA *and* FB *be also equimultiples of the same; I say then that the sum* $\overline{EA+FA}$ *will be the same multiple of* A *that the sum* $\overline{EB+FB}$ *is of* B.

For since the number of *A*s in *EA* is the same with the number of *B*s in *EB*; and since also the number of *A*s in *FA* is the same with the number of *B*s in *FB*, add equals to equals, and the number of *A*s in $\overline{EA+FA}$ will be the same with the number of *B*s in $\overline{EB+FB}$, that is, the sum $\overline{EA+FA}$ will be the same multiple of *A* that the sum $\overline{EB+FB}$ is of *B*. *Q. E. D.*

PROPOSITION. 3.

268. *If* EA *and* EB *be equimultiples of any two quantities whatever* A *and* B, *and if* 3EA *and* 3EB *be any equimultiples of* EA *and* EB; *I say then, that* 3EA *and* 3EB *will also be equimultiples of* A *and* B.

This is evident from the last proposition: for since *EA* and *EB* are equimultiples of *A* and *B*; and since *EA* and *EB* are again equimultiples of the same, it follows from that proposition, that the sum 2*EA* is the same multiple of *A* that the sum 2*EB* is of *B*: again, since 2*EA* and 2*EB* are equimultiples of *A* and *B*, and since *EA* and *EB* are other equimultiples of the same, the sum 3*EA* is the same multiple of *A* that the sum 3*EB* is of *B*; and so on *ad infinitum*. *Q. E. D.*

PROPOSITION. 4.

269. *If four quantities* A, B, C *and* D *be proportionable,* A *to* B *as* C *to* D, *and if* EA *and* EC *be any equimultiples of the first and third, and* FB *and* FD *any other equimultiples of the second and fourth; I say then that these multiples will also be proportionable, provided they be taken in the same order as the proportionable quantities whereof they are multiples; that is, that* EA *will be to* FB *as* EC *is to* FD.

For let 3*EA* and 3*EC* be any equimultiples of *EA* and *EC*, and let 2*FB* and 2*FD* be any other equimultiples of *FB* and *FD*; then since 3*EA* and 3*EC* are equimultiples of *EA* and *EC*, and since *EA* and *EC* are equimultiples of *A* and *C*, it follows from the last proposition that 3*EA* and 3*EC* are equimultiples of *A* and *C*; and for the same reason 2*FB* and 2*FD* are also equimultiples of *B* and *D*. Since then *ex hypo-*

thesi, A is to B as C is to D; and since $3EA$ and $3EC$ are equimultiples of A and C, and $2FB$ and $2FD$ are also other equimultiples of B and D, it follows from the fifth definition, that $3EA$ cannot be greater than, equal to, or less than $2FB$, but $3EC$ must also be greater than, equal to, or less than $2FD$. Again, since we have four quantities EA, FB, EC, FD, whereof $3EA$ and $3EC$ represent any equimultiples of the first and third, and $2FB$ and $2FD$ any other equimultiples of the second and fourth; and since $3EA$ cannot be greater than, equal to, or less than $2FB$, but $3EC$ must in like manner be greater than, equal to, or less than $2FD$, it follows from the fifth definition, that these four quantities EA, FB, EC, FD are proportionable; that EA is to FB as EC to FD. *Q. E. D.*

SCHOLIUM.

To this place is usually referred the inversion of proportion, (though why to this, rather than to any other, I know not;) that is, that *If four quantities be proportionable, they will also be inversely proportionable: as if* A *be to* B *as* C *is to* D, *then* B *will be to* A *as* D *to* C. For let EA and EC be any equimultiples of A and C; and let FB and FD be any other equimultiples of B and D; and first let us suppose FB to be greater than EA; then will EA be less than FB: and because A is to B as C is to D, EC will also be less than FD by the fifth definition; and therefore FD will be greater than EC: thus then we see that if FB be greater than EA, FD will also be greater than EC. And after the same manner it may be demonstrated, that if FB be equal to, or less than EA, FD in like manner will be equal to, or less than EC. Since then we have four quantities B, A, D, C, whereof FB and FD are equimultiples of the first and third, and EA and EC are other equimultiples of the second and fourth; and since FB cannot be greater than, equal to, or less than EA, but FD must accordingly be greater than, equal to, or less than EC, it follows from the fifth definition, that these four quantities B, A, D, C must be proportionable; that B must be to A as D to C. *Q. E. D.*

PROPOSITION 5.

270. *If* A *and* B *be any two homogeneous quantities, whereof* A *is the greater, and whereof* EA *and* EB *are equimultiples respectively; I say then that the difference* $\overline{EA-EB}$ *will be the same multiple of the difference* $\overline{A-B}$ *that* EA *is of* A, *or* EB *of* B.

If this be denied, let G be the same multiple of $\overline{A-B}$ that EA is of A, or EB of B; then we shall have two quantities $\overline{A-B}$ and B whose sum

ſum is A, and whereof G and EB are equimultiples reſpectively; therefore by the firſt propoſition, the ſum $\overline{G+EB}$ will be the ſame multiple of the ſum A that EB is of B: but EA is alſo the ſame multiple of A that EB is of B; therefore $\overline{G+EB}$ is the ſame multiple of A that EA is of A; therefore $\overline{G+EB}$ muſt be equal to EA; take EB from both ſides, and G will be equal to $\overline{EA-EB}$: but G was the ſame multiple of $\overline{A-B}$ that EA was of A, or EB of B; therefore $\overline{EA-EB}$ will be the ſame multiple of $\overline{A-B}$ that EA is of A, or EB of B. *Q. E. D.*

PROPOSITION 6.

271. *If from* EA *and* EB, *equimultiples of any two quantities* A *and* B, *be ſubtracted* FA *and* FB *any other equimultiples of the ſame; the remainders* $\overline{\text{EA}-\text{FA}}$ *and* $\overline{\text{EB}-\text{FB}}$ *will either be equal to the quantities* A *and* B *reſpectively, or they will be equimultiples of them.*

CASE 1.

In the firſt place, let the remainder $\overline{EA-FA}$ be equal to A; I ſay then that the other remainder $\overline{EB-FB}$ will alſo be equal to B. For ſince FA is the ſame multiple of A that FB is of B, it follows from the nature of multiples, that $\overline{FA+A}$ will be the ſame multiple of A that $FB+B$ is of B: but A is equal to $\overline{EA-FA}$; and adding FA to both ſides we have $\overline{FA+A}=EA$; therefore inſtead of ſaying as before, that $\overline{FA+A}$ is the ſame multiple of A that $\overline{FB+B}$ is of B, we may now ſay that EA is the ſame multiple of A that $\overline{FB+B}$ is of B: but EA is the ſame multiple of A that EB is of B; therefore EB is the ſame multiple of B that $\overline{FB+B}$ is of B; therefore EB is equal to $\overline{FB+B}$; ſubtract FB from both ſides, and you will have $\overline{EB-FB}=B$. *Q. E. D.*

CASE 2.

Let us now ſuppoſe the remainder $\overline{EA-FA}$ to be ſome multiple of A; for if A meaſures both EA and FA, it muſt meaſure $\overline{EA-FA}$; and ſo $\overline{EA-FA}$ muſt be ſome multiple of A; and for the ſame reaſon, the other remainder $\overline{EB-FB}$ muſt be ſome multiple of B: I ſay then

then in the next place, that $\overline{EB-FB}$ muſt be the ſame multiple of B that $\overline{EA-FA}$ is of A. If this be denied, let G be the ſame multiple of B that $\overline{EA-FA}$ is of A; then ſince $\overline{EA-FA}$ and G are equimultiples of A and B, and ſince FA and FB are alſo other equimultiples of the ſame, it follows from the ſecond propoſition, that the ſum $\overline{EA-FA+FA}$ will be the ſame multiple of A that $\overline{G+FB}$ is of B: but $\overline{EA-FA+FA}$ $=EA$; therefore EA is the ſame multiple of A that $\overline{G+FB}$ is of B: but EA is the ſame multiple of A that EB is of B; therefore EB is the ſame multiple of B that $\overline{G+FB}$ is of B; therefore EB is equal to $\overline{G+FB}$; therefore $\overline{EB-FB}$ is equal to G: but G was the ſame multiple of B that $\overline{EA-FA}$ is of A by the ſuppoſition; therefore $\overline{EB-FB}$ is the ſame multiple of B that $\overline{EA-FA}$ is of A. *Q. E. D.*

SCHOLIUM.

As in the ſecond definition it was provided that no ſimple quantity be conſidered as a multiple of itſelf, ſo in this propoſition care is taken that no two ſimple quantities be conſidered as equimultiples of themſelves; which indeed is but a conſequence of that definition, and is the reaſon why this propoſition reſolves itſelf into two caſes.

For a better underſtanding and remembering the ſtructure of the ſix foregoing propoſitions, it may be obſerved, that the two laſt propoſitions are nothing elſe but the two firſt with their ſigns changed. In the firſt propoſition it was demonſtrated, that the ſum $\overline{EA+EB}$ is the ſame multiple of the ſum $\overline{A+B}$ that EA is of A, or EB of B: in the fifth propoſition it is demonſtrated, that the difference $\overline{EA-EB}$ is the ſame multiple of the difference $\overline{A-B}$ that EA is of A or EB of B. Again, in the ſecond propoſition it was demonſtrated, that the ſum $\overline{EA+FA}$ is the ſame multiple of A that the ſum $\overline{EB+FB}$ is of B; and in the ſixth it is demonſtrated that the remainder $\overline{EA-FA}$ is the ſame multiple of A that the remainder $\overline{EB-FB}$ is of B.

PROPOSITION 7.

272. *If two equal quantities* A *and* B *be compared with a third as* C, *I ſay then, that both* A *and* B *will have the ſame proportion to* C; *and* vice verſa, *that* C *will have the ſame proportion both to* A *and to* B.

For

For taking any equimultiples of A and B, ſuppoſe $3A$ and $3B$, and any other multiple of C, ſuppoſe $5C$, it is plain that $3A$ muſt be equal to $3B$, becauſe A is equal to B: but if $3A$ be equal to $3B$, then it will be impoſſible for $3A$ to be greater than, equal to, or leſs than $5C$, but $3B$ muſt accordingly be greater than, equal to, or leſs than the ſame $5C$; therefore we have four quantities A, C, B and C, whereof $3A$ and $3B$ repreſent any equimultiples of the firſt and third, and $5C$ and $5C$ any other equimultiples of the ſecond and fourth; and ſince the firſt multiple $3A$ cannot be greater than, equal to, or leſs than the ſecond $5C$, but the third multiple $3B$ muſt accordingly be greater than, equal to, or leſs than the fourth $5C$, it follows from the fifth definition, that theſe four quantities A, C, B and C are proportionable, A to C as B to C. *Q. E. D.*

Again, ſince $3A$ is equal to $3B$, it will be impoſſible for $5C$ to be greater than, equal to, or leſs than $3A$, but the ſame $5C$ muſt alſo be greater than, equal to, or leſs than $3B$; therefore we have four quantities C, A, C and B, whereof $5C$ and $5C$ repreſent any equimultiples of the firſt and third, and $3A$ and $3B$ any other equimultiples of the ſecond and fourth; and ſince the firſt multiple $5C$ cannot be greater than, equal to, or leſs than the ſecond $3A$, but the third multiple $5C$ muſt alſo be greater than, equal to, or leſs than the fourth $3B$, it follows from the fifth definition, that theſe four quantities C, A, C and B muſt be proportionable, C to A as C to B. *Q. E. D.*

PROPOSITION 8.

273. *If two unequal quantities* A *and* B, *whereof* A *is the greater, be compared with a third as* C, *I ſay then that* A *will have a greater proportion to* C *than* B *hath to* C; *but that on the other hand,* C *will have a greater proportion to* B *than it hath to* A.

For ſince by the ſuppoſition, A is greater than B, $\overline{A-B}$ will be the exceſs of A above B; and by the fifth propoſition, if EB be any multiple of B, $\overline{EA-EB}$ will be the ſame multiple of $\overline{A-B}$: multiply then theſe two quantities B and $\overline{A-B}$ alike, till of the equimultiples thence ariſing, the leſs ſhall be greater than C; then will the other be much greater; let theſe equimultiples be $3B$ and $\overline{3A-3B}$, each being greater than C: laſtly multiply C till you come to a multiple of it that ſhall be the next greater than $3B$, which multiple let be $5C$; then it is plain that $3B$ cannot be leſs than $4C$; for if it was, then $4C$, and not $5C$ would be the next multiple of C greater than $3B$, contrary to the ſuppoſition. Since then $3B$ cannot be leſs than $4C$, it follows, that if to

3*B* be added a greater quantity, and to 4*C* a less, the former sum will be greater than the latter: but 3*A*—3*B* is greater than *C* by the construction; add then 3*A*—3*B* to 3*B*, and *C* to 4*C*, and you will have 3*A* greater than 5*C*: but 3*B* is less than 5*C* by the construction; therefore we have four quantities *A*, *C*, *B* and *C*, whereof 3*A* and 3*B* are equimultiples of the first and third, and 5*C* and 5*C* are other equimultiples of the second and fourth; and since the first multiple 3*A* is greater than the second 5*C*, and at the same time the third multiple 3*B* is not greater than the fourth 5*C*, but less, it follows from the seventh definition, that of the four quantities *A*, *C*, *B* and *C*, *A* hath a greater proportion to *C* than *B* hath to *C*. *Q. E. D.*

Again, since we have four quantities *C*, *B*, *C* and *A*, whereof 5*C* and 5*C* are equimultiples of the first and third, and 3*B* and 3*A* are other equimultiples of the second and fourth; and since the first multiple 5*C* is greater than the second 3*B*, and at the same time the third multiple 5*C* is not greater than the fourth 3*A*, but less, it follows from the seventh definition, that of the four quantities *C*, *B*, *C* and *A*, *C* hath a greater proportion to *B* than *C* hath to *A*. *Q. E. D.*

PROPOSITION 9.

274. *If two quantities* A *and* B *have both the same proportion to a third as* C, *or if* C *hath the same proportion to both* A *and* B; *in either of these cases* A *and* B *must be equal to each other.*

For should either of them be greater than the other, should *A* be greater than *B*, then by the last proposition, *A* must have a greater proportion to *C* than *B* hath to *C*, contrary to the first supposition; and *C* must have a greater proportion to *B* than it hath to *A*, contrary to the second supposition; therefore *A* and *B* must be equal to each other. *Q. E. D.*

PROPOSITION 10.

275. *If of three quantities* A, B *and* C, A *hath a greater proportion to* C *than* B *hath to* C, *or if* C *hath a greater proportion to* B *than it hath to* A; *in either of these cases* A *must be greater than* B.

For was *A* equal to, or less than *B*, then either *A* must have the same proportion to *C* that *B* hath to *C*, as in the seventh proposition, or or a less as in the eighth, both which contradict the first supposition: and again, was *A* equal to, or less than *B*, then either *C* must have the same proportion to *A* that it hath to *B*, as in the seventh proposition, or a greater as in the eighth, both which contradict the second supposition; therefore *A* must be greater than *B*. *Q. E. D.*

PRO-

PROPOSITION II.

276. *If two ratios be the same with a third, they must be the same with one another: as if the ratio of* A *to* a *and the ratio of* C *to* c *be both the same with the ratio of* B *to* b, *then the ratio of* A *to* a *will be the same with the ratio of* C *to* c: *or thus; If* A *be to* a *as* B *to* b, *and* B *to* b *as* C *to* c; *I say then that* A *will be to* a *as* C *to* c.

For taking any equimultiples of the antecedents, suppose $3A$, $3B$, $3C$; and any other equimultiples of the consequents, suppose $2a$, $2b$, $2c$, let $3A$ be greater than $2a$; then since by the supposition A is to a as B to b, and $3A$ is greater than $2a$, $3B$ must be greater than $2b$ by the fifth definition: again, since B is to b as C to c, and $3B$ is greater than $2b$, $3C$ must be greater than $2c$: thus then we see that if $3A$ be greater than $2a$, $3C$ must necessarily be greater than $2c$: and in like manner it may be demonstrated that if $3A$ be equal to, or less than $2a$, $3C$ will accordingly be equal to, or less than $2c$. Since then we have four quantities A, a, C and c, whereof $3A$ and $3C$ represent any equimultiples of the first and third, and $2a$ and $2c$ any other equimultiples of the second and fourth; and since $3A$ cannot be greater than, equal to, or less than $2a$, but $3C$ must accordingly be greater than, equal to, or less than $2c$, it follows from the fifth definition that these four quantities A, a, C and c must be proportionable, A to a as C to c. *Q. E. D.*

PROPOSITION. 12.

277. *If ever so many quantities* A, B, C *in one series be proportionable to as many* a, b, c *in another, that is,* A *to* a *as* B *to* b *as* C *to* c; *I say then, that as any one antecedent is to it's consequent, so will the sum of all the antecedents be to the sum of all the consequents; that is, as* A *is to* a *so will* $\overline{A+B+C}$ *be to* $\overline{a+b+c}$: *or if we suppose* $\overline{A+B+C}=S$, *and* $a+b+c=s$, *I say then that as* A *is to* a *so will* S *be to* s.

For taking any equimultiples of the antecedents, suppose $3A$, $3B$, $3C$, and any other equimultiples of the consequents, suppose $2a$, $2b$, $2c$, let $3A$ be greater than $2a$; then since A is to a as B to b, and $3A$ is greater than $2a$, $3B$ must be greater than $2b$ by the fifth definition: again, since B is to b as C to c, and $3B$ is greater than $2b$, $3C$ must be greater than $2c$; therefore if $3A$ be greater than $2a$, not only $3B$ will be greater than $2b$, but also $3C$ will be greater than $2c$, and consequently the whole sum $\overline{3A+3B+3C}$ will be greater than the whole sum $\overline{2a+2b+2c}$: But by the first proposition, the sum $\overline{3A+3B+3C}$ is the same mul-

 tiple

tiple of the sum $\overline{A+B+C}$ or S that $3A$ is of A; therefore $\overline{3A+3B+3C}$ $=3S$; and for the same reason $\overline{2a+2b+2c}=2s$; therefore we may now say that if $3A$ be greater than $2a$, $3S$ will be greater than $2s$: and after the same manner might it be demonstrated, that if $3A$ be equal to, or less than $2a$, $3S$ will be equal to, or less than $2s$. Since then we have four quantities A, a, S and s, whereof $3A$ and $3S$ represent any equimultiples of the first and third, and $2a$ and $2s$ any others of the second and fourth; and since $3A$ cannot be greater than, equal to, or less than $2a$, but $3S$ must in like manner be greater than, equal to, or less than $2s$, it follows from the fifth definition that these four quantities A, a, S and s must be proportionable, A to a as S to s. *Q. E. D.*

PROPOSITION 13.

278. *If* A *hath the same proportion to* a *that* B *hath to* b, *but* B *hath a greater proportion to* b *than* C *hath to* c; *I say then that* A *hath a greater proportion to* a *than* C *to* c.

For since by the supposition B is to b in a greater proportion than C to c, it follows from the seventh definition that there are equimultiples of B and C, and others again of b and c, of such a nature, that Bs multiple shall exceed that of b, and at the same time Cs multiple shall not exceed that of c: let then $3B$ exceed $2b$, and let $3C$ not exceed $2c$; then since A is to a as B to b, and $3B$ exceeds $2b$, $3A$ must necessarily exceed $2a$ by the fifth definition; therefore we have four quantities A, a, C and c, whereof $3A$ and $3C$ are equimultiples of the first and third, and $2a$ and $2c$ are other equimultiples of the second and fourth; and since $3A$ exceeds $2a$ when $3C$ does not exceed $2c$, it follows from the seventh definition that of these four quantities A, a, C and c, A hath a greater proportion to a than C hath to c. *Q. E. D.*

PROPOSITION 14.

279. *If four homogeneous quantities be proportionable, the first to the second as the third to the fourth; I say then that the second will be greater than, equal to, or less than the fourth, according as the first is greater than, equal to, or less than the third: as if* A *be to* B *as* C *is to* D; *I say then that* B *will be greater than, equal to, or less than* D, *according as* A *is greater than, equal to, or less than* C.

CASE I.

Let A be greater than C: I say then that B will be greater than D. For since A is greater than C, A will have a greater proportion to B than C hath to B by the eighth proposition: again, since C is to D

as A to B, and A hath a greater proportion to B than C hath to B, it follows from the last proposition that C is to D in a greater proportion than C to B; therefore by the tenth proposition B is greater than D. *Q. E. D.*

CASE 2.

Let now A be less than C: I say then that B will be less than D. For if A be less than C; then C will be greater than A: since then C is to D as A is to B *ex hypothesi*, and C is greater than A, it follows from the last case that D will be greater than B; and therefore B will be less than D. *Q. E. D.*

CASE 3.

Lastly let A be equal to C: I say then that B will be equal to D. For since A is equal to C, A will be to B as C is to B by the seventh proposition; but C is to D as A to B by the supposition; therefore C is to D as C is to B by the eleventh proposition; therefore B and D are equal by the ninth. *Q. E. D.*

PROPOSITION 15.

280. *Parts are in the same proportion with their respective equimultiples. Let* A *and* a *be any two homogeneous quantities, whereof* 3A *and* 3a *represent any equimultiples respectively; I say then, that* A *will be to* a *as* 3A *to* 3a.

For take B and C both equal to A, and also b and c both equal to a; then by the seventh proposition we shall have A to a as B to b as C to c; therefore by the twelfth proposition we shall have A to a as $\overline{A+B+C}$ to $\overline{a+b+c}$: but in this case $\overline{A+B+C}=3A$, and $\overline{a+b+c}=3a$; therefore A is to a as $3A$ is to $3a$. *Q. E. D.*

PROPOSITION 16.

281. *If four homogeneous quantities be proportionable, the first to the second as the third to the fourth; I say then that they will also be alternately proportionable, that is, the first to the third as the second to the fourth: as if* A *be to* B *as* C *to* D; *I say then that* A *will be to* C *as* B *to* D.

For taking any equimultiples of A and B, suppose $3A$ and $3B$, and any others of C and D, suppose $2C$ and $2D$; since $3A$ is to $3B$ as A to B by the last, and A is to B as C to D by the supposition, and C is to D as $2C$ to $2D$ by the last; it follows from the 11th proposition that $3A$ is to $3B$ as $2C$ to $2D$; therefore by the 14th proposition, $3A$

cannot be greater than, equal to, or leſs than $2C$, but at the ſame time $3B$ muſt be greater than, equal to, or leſs than $2D$. Since then we have four quantities A, C, B and D, whereof $3A$ and $3B$ repreſent any equimultiples of the firſt and third, and $2C$ and $2D$ any other equimultiples of the ſecond and fourth; and ſince $3A$ cannot be greater than, equal to, or leſs than $2C$, but $3B$ muſt accordingly be greater than, equal to, or leſs than $2D$, it follows from the fifth definition that theſe four quantities A, C, B and D muſt be proportionable, A to C as B to D. *Q. E. D.*

Note. Alternate proportion can have no place, except where all the quantities A, B, C and D are of the ſame kind: for if A and B were of one kind, and C and D of another, how would it be poſſible for the quantities A and C, or B and D to have any proportion one to another, much leſs the ſame?

PROPOSITION 17.

282. *If four quantities* A, B, C *and* D, *whereof* A *is greater than* B, *and* C *greater than* D, *be proportionable,* A *to* B *as* C *to* D; *I ſay then that* $\overline{A-B}$ *will be to* B *as* $\overline{C-D}$ *is to* D, *which is called proportion by diviſion.*

For let $3A$, $3B$, $3C$ and $3D$ be any equimultiples of the quantities A, B, C and D; then will $\overline{3A-3B}$ and $\overline{3C-3D}$ be like multiples of $\overline{A-B}$ and $\overline{C-D}$. Again, let $2B$ and $2D$ be any other equimultiples of B and D, and let $3A-3B$ be greater than $2B$; then if $3B$ be added to both ſides, we ſhall have $3A$ greater than $5B$; and becauſe A is to B as C is to D, we ſhall have by the fifth definition, $3C$ greater than $5D$; take $3D$ from both ſides, and you will have $\overline{3C-3D}$ greater than $2D$; therefore if $\overline{3A-3B}$ be greater than $2B$, $\overline{3C-3D}$ muſt be greater than $2D$: and by a like proceſs it may be demonſtrated, that if $3A-3B$ be equal to, or leſs than $2B$, $\overline{3C-3D}$ will be equal to, or leſs than $2D$. Since then we have four quantities, $\overline{A-B}$, B, $\overline{C-D}$ and D, whereof $\overline{3A-3B}$ and $\overline{3C-3D}$ repreſent any equimultiples of the firſt and third, and $2B$ and $2D$ any other equimultiples of the ſecond and fourth; and ſince $\overline{3A-3B}$ cannot be greater than, equal to, or leſs than $2B$, but at the ſame time $\overline{3C-3D}$ muſt accordingly be greater than, equal to, or leſs than $2D$, it follows from the fifth definition that theſe four quantities $\overline{A-B}$, B, $\overline{C-D}$ and D muſt be proportionable, $\overline{A-B}$ to B as $\overline{C-D}$ to D. *Q. E. D.*

PRO-

PROPOSITION 18.

283. *If four quantities* A, B, C *and* D *be proportionable,* A *to* B *as* C D; *I say then that* $\overline{A+B}$ *will be to* B *as* $\overline{C+D}$ *to* D, *which is called proportion by composition.*

If this be denied, that $A+B$ is to B as $C+D$ is to D, it must then be allowed that $A+B$ is to B as $C+D$ is to some quantity either greater or less than D; suppose to a greater, and call it E; then since E is by the supposition greater than D, if $C-E$ be added to both sides, we shall have C greater than $\overline{C+D-E}$. This being observed, let us begin again, and suppose $A+B$ to B as $C+D$ to E; then we shall have *dividendo*, (that is, by the last proposition,) $\overline{A+B-B}$ to B as $\overline{C+D-E}$ to E; but $\overline{A+B-B}$ is equal to A; therefore A is to B as $\overline{C+D-E}$ is to E; but A is to B as C is to D by the supposition; therefore C is to D as $\overline{C+D-E}$ is to E; but of these four proportionals C, D, $\overline{C+D-E}$ and E, it has been proved that the first is greater than the third, that C is greater than $\overline{C+D-E}$; therefore by the fourteenth, the second must be greater than the fourth, that is, D must be greater than E; therefore E must be less than D; therefore if $A+B$ be to B as $C+D$ is to any quantity greater than D, that quantity must also be less than D, which is impossible; therefore it is impossible for $A+B$ to be to B as $C+D$ is to any quantity greater than D: and by a like process it may be demonstrated, that it is as impossible for $A+B$ to be to B as $C+D$ is to any quantitity less than D; therefore $A+B$ must be to B as $C+D$ is to D. *Q. E. D.*

PROPOSITION 19.

284. *If from two quantities* A *and* B *in any proportion be subtracted other two* C *and* D *in the same proportion; I say then that the remainders* A—C *and* B—D *will still be in the same proportion, that is, that* A—C *will be to* B—D *as* A *to* B *or as* C *to* D.

For since by the supposition A is to B as C is to D, we shall have *permutando*, (that is, by the sixteenth proposition,) A to C as B to D; and *dividendo*, $A-C$ to C as $B-D$ to D; and again *permutando*, $A-C$ to $B-D$ as C is to D; but A is to B as C is to D; therefore $A-C$ is to $B-D$ as A to B. *Q. E. D.*

SCHOLIUM.

Here Doctor *Gregory* in his manuscript copy finds a corollary demonstrating that illation called conversion of proportion; but because it is difficult

difficult to make ſenſe of that demonſtration, I chuſe rather to inſert his own demonſtration of the ſame propoſition, which is as follows.

If four quantities A, B, C *and* D *be proportionable*, A *to* B *as* C *to* D; *I ſay then that* A *is to* A—B *as* C *is to* C—D, *which is called converſion of proportion.* For ſince by the ſuppoſition *A* is to *B* as *C* is to *D*, we ſhall have *dividendo*, *A*—*B* to *B* as *C*—*D* to *D*; and *invertendo*, *B* to *A*—*B* as *D* to *C*—*D*; and *componendo*, *B*+*A*—*B* to *A*—*B* as *D*+*C*—*D* to *C*—*D*, that is, *A* to *A*—*B* as *C* to *C*—*D*. *Q. E. D.*

As to the foregoing nineteenth propoſition I ſhall further obſerve, that as in that propoſition, by diviſion of proportion it was demonſtrated, that if from two quantities *A* and *B* in any proportion, be ſubtracted two others *C* and *D* in the ſame proportion, the remainders *A*—*C* and *B*—*D* will ſtill be in the ſame proportion with *A* and *B*; ſo by compoſition of proportion it may be demonſtrated, that if to two quantities *A* and *B* in any proportion be added two others *C* and *D* in the ſame proportion, the aggregates *A*+*C* and *B*+*D* will ſtill be in the ſame proportion with *A* and *B*; but this has already been demonſtrated, being a particular caſe of the twelfth propoſition.

PROPOSITION 20.

285. *If there be three quantities* A, B *and* C *in one ſeries, and three others* D, E *and* F *in another, and if the proportions in one ſeries be the ſame with the proportions in the other when taken in the ſame order, as if* A *be to* B *as* D *is to* E, *and* B *to* C *as* E *to* F; *I ſay then that* A *cannot be greater than, equal to, or leſs than* C *in one ſeries, but accordingly* D *muſt be greater than, equal to, or leſs than* F *in the other.*

For let *A* be greater than *C*; then it is plain from the eighth propoſition that *A* muſt have a greater proportion to *B* than *C* hath to *B*; but *A* is to *B* as *D* to *E* by the ſuppoſition, and *C* is to *B* as *F* to *E*, becauſe by the ſuppoſition *B* is to *C* as *E* to *F*; therefore *D* hath a greater proportion to *E* than *F* hath to *E*; therefore *D* is greater than *F* by the tenth propoſition; therefore if *A* be greater than *C*, *D* muſt be greater than *F*: and after the ſame manner it may be demonſtrated, that if *A* be equal to, or leſs than *C*, *D* muſt accordingly be equal to, or leſs than *F*; therefore *A* cannot be greater than, equal to, or leſs than *C*, but accordingly *D* muſt be greater than, equal to, or leſs than *F*. *Q. E. D.*

PROPOSITION 21.

286. *If there be three quantities* A, B *and* C *in one ſeries, and three others* D, E *and* F *in another, and if the proportions in one ſeries be the ſame with the proportions in the other, but in a different order,*

as

as if A *be to* B *as* E *is to* F, *and* B *to* C *as* D *is to* E; *I say still that* A *cannot be greater than, equal to, or less than* C, *but accordingly* D *must be greater than, equal to, or less than* F.

For let *A* be greater than *C*; then by the eighth proposition *A* must have a greater proportion to *B* than *C* hath to *B*; but *A* is to *B* as *E* is to *F* by the supposition, and *C* is to *B* as *E* to *D*, because by the supposition *B* is to *C* as *D* to *E*; therefore *E* hath a greater proportion to *F* than it hath to *D*; therefore *D* must be greater than *F* by the tenth proposition; therefore if *A* be greater than *C*, *D* must be greater than *F*: and by a like way of reasoning, if *A* be equal to, or less than *C*, *D* will accordingly be equal to, or less than *F*; therefore *A* cannot be greater than, equal to, or less than *C*, but accordingly *D* must be greater than, equal to, or less than *F*. *Q. E. D.*

PROPOSITION 22.

287. *If there be three quantities* A, B *and* C *in one series, and three others* D, E *and* F *in another, and if the proportions in one series be the same with the proportions in the other when taken in the same order; I say then that the extremes in one series will be in the same proportion with the extremes in the other: as if* A *be to* B *as* D *is to* E, *and* B *to* C *as* E *to* F; *I say then that* A *will be to* C *as* D *to* F.

Note. For avoiding a multiplicity of words, this consequence is said to follow *ex æquo ordinate*, or *ex æquo*: see the eighteenth and nineteenth definitions.

Take any equimultiples of *A* and *D*, suppose $4A$ and $4D$, and any others of *B* and *E*, suppose $3B$ and $3E$, and lastly any others of *C* and *F*, as $2C$ and $2F$; then since by the supposition *A* is to *B* as *D* is to *E*, it follows from the fourth proposition that $4A$ will be to $3B$ as $4D$ to $3E$: again, since by the supposition *B* is to *C* as *E* to *F*, it follows from the same fourth proposition that $3B$ will be to $2C$ as $3E$ to $2F$: so that we have three quantities, to wit $4A$, $3B$, $2C$ in one series, and three others, to wit $4D$, $3E$ and $2F$ in another; and it has been shewn that the proportions in one series are the same with the proportions in the other when taken in the same order, that is, $4A$ is to $3B$ as $4D$ to $3E$, and $3B$ to $2C$ as $3E$ to $2F$; therefore by the twentieth proposition, $4A$ cannot be greater than, equal to, or less than $2C$, but $4D$ must accordingly be greater than, equal to, or less than $2F$. Since then we have four quantities *A*, *C*, *D* and *F*, whereof $4A$ and $4D$ represent any equimultiples of the first and third, and $2C$ and $2F$ any other equimultiples of the second and fourth; and since $4A$ cannot be greater than, equal to, or less

less than $2C$, but accordingly $4D$ must be greater than, equal to, or less than $2F$, it follows from the fifth definition that these four quantities A, C, D and F are proportionable, A to C as D to F. *Q. E. D.*

COROLLARY.

In like manner if there be ever so many quantities A, B, C, G, &c *in one series, and as many others* D, E, F, H, &c *in another, and if* A *be to* B *as* D *is to* E, *and* B *to* C *as* E *to* F, *and* C *to* G *as* F *to* H, &c, *the consequence with respect to the extremes will still be the same, that is,* A *will be to* G *as* D *to* H: for it has been proved already that A is to C as D to F; and by the supposition C is to G as F to H; therefore *ex æquo*, A will be to G as D to H.

PROPOSITION 23.

288. *If there be three quantities* A, B *and* C *in one series, and three others* D, E *and* F *in another, and if the proportions in one series be the same with the proportions in the other, but in a different order; I say that the extremes in one series will still be in the same proportion with the extremes in the other: as if* A *be to* B *as* E *is to* F, *and* B *to* C *as* D *to* E; *I say still that* A *will be to* C *as* D *to* F.

Note. This consequence is said to be *ex æquo perturbate.*

Take any equimultiples of A, B and D, suppose $3A$, $3B$ and $3D$, and any others of C, E and F, suppose $2C$, $2E$ and $2F$, and the reasoning is as follows: $3A$ is to $3B$ as A to B by the fifteenth, and A is to B as E to F by the supposition, and E is to F as $2E$ to $2F$ by the fifteenth; therefore $3A$ is to $3B$ as $2E$ to $2F$ by the eleventh: again, B is to C as D to E by the supposition; therefore $3B$ will be to $2C$ as $3D$ to $2E$ by the fourth: since then we have three quantities, to wit $3A$, $3B$ and $2C$ in one series, and three others, to wit $3D$, $2E$ and $2F$ in another, and since the proportions are the same in both serieses, but in a different order, that is, since $3A$ is to $3B$ as $2E$ to $2F$, and $3B$ is to $2C$ as $3D$ to $2E$, it follows from the twentyfirst proposition, that $3A$ cannot be greater than, equal to, or less than $2C$, but $3D$ must accordingly be greater than, equal to, or less than $2F$: again, since we have four quantities A, C, D and F, whereof $3A$ and $3D$ represent any equimultiples of the first and third, and $2C$ and $2F$ any others of the second and fourth, and since $3A$ cannot be greater than, equal to, or less than $2C$, but $3D$ must accordingly be greater than, equal to, or less than $2F$, it follows from the fifth definition that these four quantities A, C, D and F are proportionable, A to C as D to F. *Q. E. D.*

PROPOSITION 24.

289. *If there be six quantities* A, B, C, D, E, F, *whereof* A *is to* B *as* C *is to* D, *and* E *is to* B *as* F *to* D; *I say then that* A+E *will be to* B *as* C+F *to* D.

For since by the supposition *E* is to *B* as *F* to *D*, we shall have *invertendo*, *B* to *E* as *D* to *F*. Since then *A* is to *B* as *C* is to *D* by the supposition, and *B* is to *E* as *D* to *F*, it follows *ex æquo*, that *A* is to *E* as *C* to *F*; whence *componendo*, *A+E* will be to *E* as *C+F* is to *F*: again, since *A+E* is to *E* as *C+F* is to *F*, and *E* is to *B* as *F* to *D* by the supposition, it follows again *ex æquo*, that *A+E* is to *B* as *C+F* to *D*. *Q. E. D.*

LEMMA.

290. *If four quantities* A, B, C *and* D *be proportionable*, A *to* B *as* C *to* D; *I say then that* A *cannot possibly be greater than, equal to, or less than* B, *but that* C *will accordingly be greater than, equal to, or less than* D.

That this lemma is self-evident according to the common notion of proportionality, or even upon the plan of the fifth definition, were simple quantities allowed to be considered as equimultiples of themselves, is what I suppose will scarce be denied: but this the name of multiple and equimultiple will by no means admit of, and therefore care has been taken to provide against it, as may be seen in my observations on the second definition, and at the end of the sixth proposition: therefore as the doctrine of proportion here stands, this lemma ought certainly to be demonstrated; and the author's taking it for granted in the demonstration of the next proposition following, where he might with so much ease have avoided it, is not so much an argument of it's self-evidency, as that he had demonstrated it somewhere before in this fifth book, but that it is now lost. *Commandine*, from the fourteenth of this book, has demonstrated one particular case of this proposition, that is, where the quantities *A*, *B*, *C* and *D* are all of a kind; but this proposition is no less true when the quantities *A* and *B* are of one kind, and *C* and *D* of another. This *Clavius* very well observes, and endeavours to demonstrate this proposition in this more extended sense; (see his scholium to the fourteenth proposition of the fifth book;) but whether this demonstration of his amounts to any more than proving *idem per idem*, let them that read it judge. The demonstration I shall here give of it is as follows.

I am to demonstrate that if *A* be to *B* as *C* is to *D*; then *A* cannot possibly be greater than, equal to, or less than *B*, but accordingly *C* must be greater than, equal to, or less than *D*.

 CASE

CASE 1.

Let A be greater than B; I fay then that C muft be greater than D. For fince A is greater than B, multiply the excefs $A-B$ to a multiple greater than B, and let this multiple be $\overline{3A-3B}$; then fince $\overline{3A-3B}$ is greater than B, if $3B$ be added to both fides, we fhall have $3A$ greater than $4B$: again, fince A is to B as C is to D, and $3A$ is greater than $4B$, we fhall have by the fifth definition, $3C$ greater than $4D$; therefore $3C$ muft be much greater than $3D$, and C muft be greater than D. *Q. E. D.*

CASE 2.

Let now A be lefs than B; I fay then that C muft be lefs than D. For fince A is to B as C is to D, we fhall have *invertendo*, B to A as D to C; but B is greater than A, becaufe by the fuppofition A is lefs than B; therefore D muft be greater than C by the laft cafe; therefore C muft be lefs than D. *Q. E. D.*

CASE 3.

Laftly let A be equal to B; I fay then that C muft be equal to D. For fince C is to D as A is to B, fhould C be greater or lefs than D, A would accordingly be greater or lefs than B by the two laft cafes; but A is neither greater nor lefs than B by the fuppofition; therefore C is neither greater nor lefs than D; therefore C is equal to D. *Q. E. D.*

PROPOSITION 25.

291. *If four quantities* A, B, C *and* D *be proportionable*, A *to* B *as* C *to* D; *I fay then that the fum of the greateft and leaft terms put together will be greater than the fum of the other two.*

Let A be the greateft of all; then fince A is to B as C is to D, and B is lefs than A, D will be lefs than C by the lemma: again, fince A is to B as C is to D, and C is lefs than A, D will be lefs than B by the fourteenth; therefore if A be the greateft of all, D, which is lefs than either A, B or C, will be the leaft of all, and fo the fum of the greateft and leaft terms added together will be $A+D$; therefore the fum of the other two will be $B+C$. We are now then to prove that the fum $A+D$ is greater than the fum $B+C$, which is thus done: It has been demonftrated in the nineteenth propofition, that if from two quantities A and B in any proportion whatever, be fubtracted other two C and D in the fame proportion, the remainder $A-C$ will be to the remainder $B-D$ as A to B; but A is greater than B by the fuppofition; therefore $A-C$ muft be greater than $B-D$ by the lemma; add $C+D$ to both fides, and you will have $A+D$ greater than $B+C$. *Q. E. D.*

COROL-

COROLLARY.

If three quantities A, B *and* C *be in continual proportion,* A *to* B *as* B *to* C; *I say then that the sum of the extremes will be greater than twice the middle term, that* $A+C$ *will be greater than* $B+B$ *or* 2B.

OF THE COMPOSITION AND RESOLUTION OF RATIOS.

N. B. As numbers are quantities whereof we have more diſtinct ideas than of any other quantities whatever, and as all ratios muſt be reduced to thoſe of numbers before we can make any conſiderable uſe of their compoſition and reſolution in computing the quantities of time, ſpace, velocity, motion, force, *&c*; I ſhall confine myſelf chiefly to this ſort of ratios in what I have to deliver in the following articles.

DEFINITION I.

292. *In comparing ratios, that ratio is ſaid to be greater than, equal to, or leſs than another, whoſe antecedent hath a greater, or an equal, or a leſs proportion to it's conſequent than the other's antecedent hath to it's conſequent.* Thus the ratio of 6 to 3 is ſaid to be greater, and the ratio of 4 to 3 leſs than the ratio of 5 to 3: thus again, the ratio of 6 to 3 is ſaid to be greater, and the ratio of 6 to 5 leſs than the ratio of 6 to 4, *&c.* *Therefore whenever two ratios are to be compared whoſe antecedents and conſequents are both different, it will be proper to reduce them to the ſame antecedent or to the ſame conſequent, before the compariſon is made.* As for inſtance; ſuppoſe any one would know which of theſe two ratios is the greater, to wit, the ratio of 7 to 5, or the ratio of 4 to 3: to know this, it will be proper to ſet off one of the ratios, ſuppoſe that of 4 to 3, from 7 the antecedent of the other, (by which phraſe I mean no more than finding a number to which 7 hath the ſame proportion that 4 hath to 3;) and this may be done by ſaying, as 4 is to 3 ſo is 7 to $\frac{21}{4}$ or $5\frac{1}{4}$: thus then it appears that the proportion of 4 to 3 is the ſame with the proportion of 7 to $5\frac{1}{4}$; ſo that now the queſtion turns upon this, which of theſe two ratios is the greater, that of 7 to 5, or that of 7 to $5\frac{1}{4}$? and the anſwer is ready, to wit, that the ratio of 7 to 5 is the greater ratio, by the eighth propoſition of the fifth book of the elements; therefore the ratio of 7 to 5 is greater than the ratio of 4 to 3. Again, ſuppoſe I would compare the ratio of 3 to 4 with the ratio of 5 to 7; then I would ſet off the ratio of 3 to 4 from 5, by ſaying, as 3 is to 4 ſo is

5 to $\frac{20}{3}$ or $7-\frac{1}{3}$; whereby it appears that the ratio of 3 to 4 is the fame with the ratio of 5 to $7-\frac{1}{3}$; but the proportion of 5 to $7-\frac{1}{3}$ is greater than the proportion of 5 to 7, as is evident from the eighth propofition of the fifth book of the elements, and alfo from the very nature of ratios, the number 5 having more magnitude when compared with $7-\frac{1}{3}$ than it hath when compared with 7; therefore the ratio of 3 to 4 is greater than the ratio of 5 to 7.

There is alfo another way of comparing ratios by turning their terms into fractions, making the antecedents numerators, and the confequents denominators. Thus the ratio of A to B is greater than, equal to, or lefs than the ratio of C to D, according as the fraction $\frac{A}{B}$ is greater than, equal to, or lefs than the fraction $\frac{C}{D}$: for the ratio of $\frac{A}{B}$ to 1 is greater than, equal to, or lefs than the ratio of $\frac{C}{D}$ to 1, according as the fraction $\frac{A}{B}$ is greater than, equal to, or lefs than the fraction $\frac{C}{D}$; this is evident from what has been laid down already: but the ratio of $\frac{A}{B}$ to 1 is the fame with the ratio of A to B, and the ratio of $\frac{C}{D}$ to 1 is the fame with the ratio of C to D; therefore the ratio of A to B is greater than, equal to, or lefs than the ratio of C to D according as the fraction $\frac{A}{B}$ is greater than, equal to, or lefs than the fraction $\frac{C}{D}$. But this way of reprefenting ratios by fractions, though it may ferve well enough for comparing them as to greater and lefs, yet ought it not by any means to be admitted in general, becaufe thefe reprefentatives are not in the fame proportion with the ratios reprefented by them: thus the fraction $\frac{6}{2}$ is double of the fraction $\frac{3}{2}$, but yet it muft by no means be concluded from thence that the ratio of 6 to 2 is double of the ratio of 3 to 2; for it will be found hereafter that the ratio of 9 to 4 is double of the ratio of 3 to 2. For my own part, I never was a favourer of reprefenting ratios by fractions, or even fraction-wife, as is done by *Barrow* and others; not only for the reafons above given, but alfo becaufe that this way of reprefenting ratios is very apt to miflead beginners into wrong conceptions of their compofition and refolution.

DEFI-

DEFINITION 2.

293. *In a ſeries of quantities of any kind whatſoever increaſing or decreaſing from the firſt to the laſt, the ratio of the extremes is ſaid to be compounded of all the intermediate ratios.* Thus if A, B, C, D repreſent any number of quantities put down (or imagined to be put down) in a ſeries, the ratio of A to D is ſaid to be compounded of, or to be reſolvable into theſe ratios, to wit, the ratio of A to B, the ratio of B to C, and the ratio of C to D: or thus; *If* A *and* D *be any two quantities, and if* B, C, &c *repreſent any number of other intermediate quantities interpoſed at pleaſure between* A *and* D, *the ratio of* A *to* D *is ſaid by this interpoſition to be reſolved into the ratios of* A *to* B, *of* B *to* C, *and of* C *to* D.

$$A, \quad B, \quad C, \quad D.$$
$$48, \quad 40, \quad 30, \quad 15.$$

This is no propoſition to be proved, but a definition laid down of what Mathematicians commonly mean by the compoſition and reſolution of ratios, which is certainly no more than what they mean by compoſition and reſolution in the caſe of any other *continuum* whatever. As for inſtance; ſuppoſe the letters A, B, C, D inſtead of repreſenting quantities, to repreſent ſo many diſtinct points placed in a right line one after another, whether at equal or unequal diſtances it matters not: who then would ſcruple to ſay that the whole interval AD conſiſted of the intervals AB, BC, CD, as of it's parts? Or if the points A and D be the extremities of a line, and any number of points B, C, &c be marked at pleaſure upon it; who will not ſay that the line AD is by theſe points reſolved or diſtinguiſhed into the parts AB, BC, CD, &c? This is the caſe in the compoſition and reſolution of lines; and I ſee no difference when applied to the compoſition and reſolution of ratios, except that here the whole and all it's parts are lines, and there the whole and all it's parts are ratios.

If A, B, C, D, &c ſignify quantities, the ratio of A to B begins at A and terminates in B; the ratio of B to C begins at B where the former left off, and terminates in C; and the ratio of C to D begins at C and terminates in D: why then ſhould not theſe continued ratios be conceived as parts conſtituting the whole ratio of A to D? That ratios are capable of being compared as to greater and leſs, and that one ratio may be greater than, equal to, or leſs than another, we have ſeen already; and if ſo, why ſhould not ratios be allowed to have quantity as well as all other things that are capable of being ſo compared? but if ratios have quantity, they muſt have parts, and theſe parts muſt be of the ſame nature with the whole, becauſe ratios are not capable of being compared with any thing but ratios: therefore I do not ſee but that the idea I have here given of the compoſition and reſolution of ratios is as juſt and

as

as intelligible as it is when applied to any other compoſition or reſolution whatſoever.

To proceed then, let *A*, *B*, *C*, *D* be points in a right line as before; let the line *AB* be equal to any line *Rr*, let *BC* be equal to ſome other line *Ss*, and *CD* to the line *Tt*; then it will not only be proper to ſay that the line *AD* is equal to the three lines *AB*, *BC*, *CD*, but alſo that the ſame line *AD* is equal to the three lines *Rr*, *Ss* and *Tt* put together: and the ſame conſideration is ſtill applicable to ratios; for ſuppoſing *A*, *B*, *C*, *D* again to ſignify quantities, as alſo *R*, *S*, *T*, *r*, *s*, *t*; let *A* be to *B* as *R* to *r*, let *B* be to *C* as *S* to *s*, and let *C* be to *D* as *T* to *t*; then it is uſual amongſt Mathematicians not only to conſider the ratio of *A* to *D* as compounded of the leſſer ratios of *A* to *B*, of *B* to *C* and of *C* to *D*, but alſo as compounded of the ratios of *R* to *r*, of *S* to *s*, and of *T* to *t*. All this will be very intelligible, if we attend to the ſeries already deſcribed; for there the ratio of 48 to 15 was compounded of the ratio of 48 to 40, of 40 to 30, and of 30 to 15; but 48 is to 40 as 6 to 5, and 40 is to 30 as 4 to 3, and 30 is to 15 as 2 to 1; therefore it is as proper to conſider the ratio of 48 to 15 as compounded of the ratios of 6 to 5, of 4 to 3, and of 2 to 1, as it is to conſider it as compounded of the ratios of 48 to 40, of 40 to 30, and of 30 to 15.

DEFINITINON 3.

294. *As when a line is divided into any number of equal parts, the whole line is ſaid to be ſuch a multiple of any one of theſe parts as is expreſſed by the number of parts into which the whole is ſuppoſed to be divided; ſo in a ſeries of continual proportionals, where the intermediate ratios are all equal to one another, and conſequently to ſome common ratio that indifferently repreſents them all, the ratio of the extremes is ſaid to be ſuch a multiple of this common ratio as is expreſſed by the number of ratios from one extreme to the other.* Thus 9, 6 and 4 are continual proportionals, whoſe common ratio is that of 3 to 2; for 9 is to 6 as 3 to 2, and 6 is to 4 as 3 to 2; therefore in this caſe, the ratio of 9 to 4 is ſaid to be the double of the ratio of 3 to 2; and on the other hand, the ratio of 3 to 2 is ſaid to be the half of the ratio of 9 to 4; but the common expreſſion is, that 9 is to 4 in a duplicate ratio of 3 to 2, and 3 is to 2 in a ſubduplicate ratio of 9 to 4. Again, 27, 18, 12 and 8 are in continual proportion, whoſe common ratio is that of 3 to 2; therefore 27 is to 8 in a triplicate ratio of 3 to 2, and 3 is to 2 in a ſubtriplicate ratio of 27 to 8. Laſtly 81, 54, 36, 24 and 16 are continual proportionals, whoſe common ratio is that of 3 to 2; therefore 81 is to 16 in a quadruplicate ratio of 3 to 2, and 3 is to 2 in a ſubquadruplicate ratio of 81 to 16. By theſe

inſtances we ſee that one ratio may not only be greater or leſs than another, but a multiple, or an aliquot part of another; nay there is no proportion can be aſſigned which ſome one ratio may not have to another: thus the ratio of 81 to 16 is found to be to the ratio of 27 to 8, as 4 to 3, becauſe the former ratio contains the ratio of 3 to 2 four times, and the latter three times: thus again, the ratio of 27 to 8 is to the ratio of 9 to 4, as 3 to 2, becauſe the former contains the ratio of 3 to 2 three times, and the latter twice; whereby it appears that proportion is competible even to ratios themſelves, as well as to all other continued quantities whatever. But though all ratios are in ſome certain proportion one to another, yet this proportion cannot always be expreſſed; I mean when the quantities of ratios are incommenſurable to one another; for ratios may be incommenſurable as well as any other continued quantities of what kind ſoever: thus the ratio of 4 to 3 is incommenſurable to the ratio of 3 to 2; which is the caſe of moſt ratios, though not of all. If all ratios were commenſurable to one another, their logarithms would be ſo too; and ſo the logarithms of all the natural numbers might be accurately aſſigned; whereas from other principles we know to the contrary, as will be ſeen when we come to treat particularly of logarithms.

N. B. The beſt way of repreſenting the quantities of ratios that I know of, is by *Gunter*'s line, where as many of the natural numbers as can be placed upon it are diſpoſed, not at equal diſtances one from another, but at diſtances proportionable to the ratios they are in one to another. Thus the diſtance between 1 and 2 is equal to the diſtance between 2 and 4, becauſe the ratio of 1 to 2 is equal to the ratio of 2 to 4: thus again, the diſtance between 4 and 9 is double the diſtance between 2 and 3, becauſe the ratio of 4 to 9 is double the ratio of 2 to 3; and ſo of the reſt.

Of the addition of ratios.

295. Since all ratios are quantities, as has been ſhewn in the three laſt articles, it follows, that they alſo as well as all other quantities muſt be capable of addition, ſubtraction, multiplication and diviſion: to treat then of theſe operations in their order, I ſhall begin firſt with addition.

If the ratios to be added be continued ratios, that is, if they lie in a ſeries wherein the antecedent of every ſubſequent ratio is the ſame with the conſequent of the ratio that went immediately before, their addition is beſt performed by throwing out all the intermediate terms: thus the ratios of A to B, of B to C and of C to D when added together, make up the ratio of A to D, as was ſhewn in the 293d article.

Therefore if the ratios to be added be diſcontinued, it will be proper to continue

continue them from some given antecedent, suppose from unity, before they can be added, thus: let the ratio of A to B, the ratio of C to D, and the ratio of E to F be proposed to be added into one sum: now the ratio of A to B set off from 1 reaches to $\frac{B}{A}$, because A is to B as 1 to $\frac{B}{A}$; the next ratio of C to D set off from $\frac{B}{A}$ reaches to $\frac{BD}{AC}$; and the last ratio of E to F set off from $\frac{BD}{AC}$ reaches to $\frac{BDF}{ACE}$; therefore the ratios of A to B, of C to D and of E to F when added together, make the ratio of 1 to $\frac{BDF}{ACE}$, which is the same with the ratio of ACE to BDF; whence we have the following canon:

Multiply first the antecedents of all the ratios proposed together, and then their consequents, and the ratio of the products thence arising will be the sum of the ratios proposed.

That the ratios of A to B, of C to D and of E to F all together constitute the ratio of ACE to BDF may be further confirmed by setting them off from ACE and from one another thus: the ratio of A to B set off from ACE reaches to BCE; in the next place the ratio of C to D set off from BCE reaches to BDE; and lastly the ratio of E to F set off from BDE reaches to BDF; therefore all these ratios together constitute the ratio of ACE to BDF. An example in numbers take as follows: let it be required to add these four ratios together, *viz.* the ratio of 2 to 3, the ratio of 4 to 5, the ratio of 6 to 7 and the ratio of 8 to 9. Here the product of the antecedents is $2 \times 4 \times 6 \times 8 = 384$, and the product of the consequents is $3 \times 5 \times 7 \times 9 = 945$; therefore the sum of all the ratios proposed is the ratio of 384 to 945; and the proof is easy: for the ratio of 2 to 3 reaches from 384 to 576; the ratio of 4 to 5 reaches from 576 to 720; the ratio of 6 to 7 reaches from 720 to 840; and the ratio of 8 to 9 reaches from 840 to 945; therefore the ratios of 2 to 3, of 4 to 5, of 6 to 7, and of 8 to 9 reach from 384 to 945.

From what has here been said concerning the addition of ratios, may easily be understood an expression so frequent among Mechanical and Philosophical writers; as when they say that A is to B in a ratio compounded of the ratio of C to D, and of the ratio of E to F; whereby they mean no more than that the ratio of A to B is equal to the sum of the ratios of C to D, and of E to F; or that A is to B as CE to DF.

According to the Mathematicians, every ratio is either a *ratio majoris inæqualitatis*, or a *ratio æqualitatis*, or a *ratio minoris inæqualitatis*, which takes in all sorts of ratios: for by a *ratio majoris inæqualitatis* they

they mean the ratio that any greater quantity hath to a less; by a *ratio minoris inæqualitatis* they mean the contrary, that is, the ratio of a lesser quantity to a greater; and therefore by a *ratio æqualitatis* they mean the ratio (if it may be called so) that every quantity hath to it's equal. If we distinguish ratios according to the effects they have in composition, then every *ratio majoris inæqualitatis* ought to be looked upon as affirmative, because such ratios always increase those to which they are added; on the other hand, the *rationes minoris inæqualitatis* ought to be considered as negative, because these always diminish the ratios to which they are added; therefore the *ratio æqualitatis* ought to be looked upon as having no magnitude at all, because such ratios have no effect in composition. Thus if to the ratio of 5 to 3 be added the ratio of 3 to 2, the sum will be the ratio of 5 to 2, as above; but the ratio of 5 to 2 is greater than the ratio of 5 to 3; therefore the ratio of 3 to 2 ought to be looked upon as affirmative, because it increases the ratio to which it is added: on the other hand, if to the ratio of 5 to 3 be added the ratio of 3 to 4, the sum will be the ratio of 5 to 4, which is less than the ratio of 5 to 3, and therefore the ratio of 3 to 4 is negative: lastly, if to the ratio of 5 to 3 be added the ratio of 3 to 3, the sum will still be the ratio of 5 to 3; therefore the ratio of 3 to 3 is nothing.

Whenever a ratio is to be resolved into two others by any arbitrary interposition of an intermediate term, it may be thought however that this intermediate term should be some intermediate magnitude between the terms of the ratio to be resolved; and so we supposed it in the 293d article: but that restriction was only supposed to prevent unseasonable objections that might otherwise arise about it; for there is no necessity that the intermediate term should be of an intermediate magnitude betwixt the extremes if we allow of negative ratios; for the ratio of 5 to 4 (for instance) may be resolved into the two ratios of 5 to 3 and of 3 to 4, though the intermediate term 3 be out of the limits of 5 and 4. This I say is plain; for though the ratio of 5 to 3, which is one of the parts, be greater than the ratio of 5 to 4, yet the ratio of 3 to 4, which is the other part, is negative, and qualifies the other in the composition, so as to reduce it to the ratio of 5 to 4: so 9 may be looked upon as a part of 7, provided the other part be —2.

COROLLARY.

If there be a series of quantities A, B, C, D, *whereof* A *is to* B *as* R *to* r, *and* B *is to* C *as* S *to* s, *and* C *is to* D *as* T *to* t; *I say then that* A *will be to* D *as* RST *the product of all the antecedents, to* rst *the product of all the consequents.* For by art. 293 the ratio of A to D is com-

pounded of the ratios of R to r, of S to s, and of T to t; and these ratios, when thrown into one sum, constitute the ratio of RST to rst; therefore A is to D as RST to rst.

Of the subtraction of ratios.

296. *The subtraction of ratios one from another, when both have the same antecedent, or both the same consequent, is obvious enough: thus the ratio of* A *to* B *subtracted from the ratio of* A *to* C *leaves the ratio of* B *to* C; *and the ratio of* B *to* C *subtracted from the ratio of* A *to* C *leaves the ratio of* A *to* B: this I say is obvious, because (according to art. 293) the ratio of A to C contains the ratios of A to B and of B to C; and therefore if either part be taken away, there must remain the other.

But if the two ratios, whereof one is to be subtracted from the other, have neither the same antecedent nor the same consequent, it will be proper then to reduce them to the same antecedent, by setting off the ratio to be subtracted from the antecedent of the other, thus: let it be required to subtract the ratio of C to D from the ratio of A to B: now the ratio of C to D set off from A reaches to $\frac{AD}{C}$; therefore to subtract the ratio of C to D from the ratio of A to B is the same as to subtract the ratio of A to $\frac{AD}{C}$ from the ratio of A to B; but the ratio of A to $\frac{AD}{C}$ subtracted from the ratio of A to B, a ratio of the same antecedent, leaves the ratio of $\frac{AD}{C}$ to B, or of AD to BC; therefore the ratio of C to D subtracted from the ratio of A to B leaves the ratio of AD to BC. The rule then is as follows:

Whenever one ratio is to be subtracted from another, change the sign of the ratio to be subtracted by inverting it's terms, and then the sum of this new ratio added to the other will be the same with the remainder of the intended subtraction. Thus to subtract the ratio of C to D from the ratio of A to B is the same as to add the ratio of D to C to the ratio of A to B; but the ratio of D to C added to the ratio of A to B gives the ratio of AD to BC by the last article; therefore the ratio of C to D subtracted from the ratio of A to B leaves the ratio of AD to BC. For a further proof of this we are to take notice, that in all subtraction whatever, the remainder and the quantity subtracted ought both together to make the quantity from whence the subtraction was made; but in our case the remainder was the ratio of AD to BC, and the quantity subtracted

tracted was the ratio of C to D, and these two added together make the ratio of ACD to BCD, or of A to B, which is the ratio from whence the subtraction was made; therefore the remainder in this case was rightly assigned.

For an example in numbers, let it be required to subtract the ratio of 4 to 5 from the ratio of 2 to 3: now the ratio of 5 to 4 added to the ratio of 2 to 3 gives the ratio of 10 to 12, or of 5 to 6, by the last article; therefore the ratio of 4 to 5 subtracted from the ratio of 2 to 3 leaves the ratio of 5 to 6, which may be confirmed thus: the ratio of 2 to 3 is the same with the ratio of 4 to 6, which contains the ratios of 4 to 5 and of 5 to 6; therefore if the ratio of 4 to 5 be taken away, there will remain the other part, which is the ratio of 5 to 6.

Before I conclude this article, I ought to take notice that there is another way of conceiving the subtraction of ratios, which for it's use in Physics and Mechanics ought not to be passed by in this place; it is thus: the ratio of C to D added to the ratio of A to B constitutes the ratio of AC to BD; therefore *e converso*, the ratio of C to D subtracted from the ratio of A to B must leave the ratio of $\frac{A}{C}$ to $\frac{B}{D}$, because multiplication and division are as much the reverse of one another as addition and subtraction; but this ratio of $\frac{A}{C}$ to $\frac{B}{D}$, when reduced to integral terms, is the same with the ratio of AD to BC found before.

N. B. *Wherever it is said that the ratio of* A *to* B *is compounded of the direct ratio of* C *to* D, *and of the inverse or reciprocal ratio of* E *to* F, *the meaning is, that the ratio of* A *to* B *is equal to the excess of the ratio of* C *to* D *above the ratio of* E *to* F, *or that* A *is to* B *as* $\frac{C}{E}$ *to* $\frac{D}{F}$, *or as* CF *to* DE.

Of the multiplication and division of ratios.

297. If the ratio of A to B be added to itself, that is, to the ratio of A to B, the sum will be the ratio of A^2 to B^2 by the last article but one; and this being added again to the ratio of A to B gives the ratio of A^3 to B^3, and so on; therefore the ratio of A^2 to B^2 is double, and the ratio of A^3 to B^3 triple of the ratio of A to B. And universally, *The ratio of* A^n *to* B^n *is such a multiple of the ratio of* A *to* B *as is expressed by the number* n. Thus the ratio of A^4 to B^4 is four times the ratio of A to B, which I prove thus: the ratio of A to B reaches first from A^4 to A^3B, 2dly from A^3B to A^2B^2, 3dly from A^2B^2 to AB^3, and 4thly from AB^3 to B^4.

To give an example in numbers, I ſay that five times the ratio of 2 to 3 is the ratio of the fifth power of 2 to the fifth power of 3, that is, the ratio of 32 to 243: for the ratio of 2 to 3 reaches 1ſt from 32 to 48, 2dly from 48 to 72, 3dly from 72 to 108, 4thly from 108 to 162, and 5thly from 162 to 243. Thus much for multiplication.

Diviſion is the reverſe of multiplication; and therefore as every ratio is doubled or tripled or quadrupled by ſquaring or cubing or ſquare-ſquaring it's terms, ſo every ratio is biſected or triſected or quadriſected by extracting the ſquare or cube or ſquare-ſquare roots of it's terms. Thus half the ratio of 2 to 3 is the ratio of the ſquare root of 2 to the ſquare root of 3, that is, (when reduced according to the firſt ſcholium in art. 179) the ratio of 40 to 49 nearly; which is further proved thus: the ratio of 40 to 49 is half the ratio of 1600 to 2401, by what was delivered in the former part of this article; but 1600 is to 2400 as 2 to 3; therefore 1600 is to 2401 as 2 to 3 very near.

But there is no neceſſity of a double extraction of the root in the diviſion of a ratio, provided the ratio propoſed be reduced to an equal one whoſe antecedent is unity. Thus 2 is to 3 as 1 to $\frac{3}{2}$, and therefore half the ratio of 2 to 3 is the ratio of 1 to $\sqrt{\frac{3}{2}}$, or the ratio of 1 to $\sqrt{1.5}$.

From what has been ſaid it appears that one ratio may be commenſurate to another, and yet the terms of one incommenſurate to the terms of the other: thus the ratio of 2 to 3 is certainly commenſurate to the ratio of the ſquare root of 2 to the ſquare root of 3, the former being double of the latter; and yet 2 and 3, the terms of the former ratio are incommenſurate to $\sqrt{2}$ and $\sqrt{3}$ the terms of the latter.

Note. *Wherever it is ſaid that* A *is to* B *in a ſeſquiplicate ratio of* C *to* D; *the meaning is, that the ratio of* A *to* B *is equal to* $\frac{3}{2}$ *of the ratio of* C *to* D: therefore in ſuch a caſe, twice the ratio of A to B will be equal to three times the ratio of C to D; but twice the ratio of A to B is the ratio of A^2 to B^2, and three times the ratio of C to D is the ratio of C^3 to D^3; therefore if A be to B in a ſeſquiplicate ratio of C to D, A^2 will be to B^2 as C^3 to D^3. Thus in the revolutions of the primary planets about the Sun, and of the ſecondary planets about Jupiter and Saturn, their periodic times are ſaid to be in a ſeſquiplicate ratio of their middle diſtances, that is, the ſquares of their periodic times are as the cubes of their middle diſtances.

Another way of multiplying and dividing ſmall ratios, that is, whoſe terms are large in compariſon of their difference.

298. Before I deliver what I have to ſay upon this head, I ſhall only obſerve, that *If two indeterminate quantities have always the ſame diffe-*

rence,

rence, the greater the quantities are, the nearer will their ratio approach towards a ratio of equality: thus the difference betwixt 2 and 1 is the same with the difference betwixt 100 and 99; but the ratio of 2 to 1 or of 100 to 50 is much greater than the ratio of 100 to 99. By the help of this observation, and the following theorem, I shall endeavour to shew that small ratios may sometimes be doubled, or tripled, or bisected, or trisected by more compendious ways than those that are taught in the last article; and whenever they happen to be so, they ought to be used, and frequently are used rather than the other.

A Theorem.

If there be two quantities whose difference is but small in comparison of the quantities themselves, and if so much be added to one and subtracted from the other as shall make their difference double, or triple, or half, or a third part of what it was before; I say then that the quantities after this alteration shall be in a duplicate, or a triplicate, or a subduplicate, or a subtriplicate ratio of that they were in before any such change was made, nearly.

1st. Let there be two numbers 10 and 11, whose difference is 1; then if $\frac{1}{2}$ be added to 11 and subtracted from 10, we shall have the numbers $11\frac{1}{2}$ and $9\frac{1}{2}$, whose difference is 2: I say now that $11\frac{1}{2}$ is to $9\frac{1}{2}$ in a duplicate ratio of 11 to 10 nearly. For the ratio of $11\frac{1}{2}$ to $9\frac{1}{2}$ is resolvable into these two ratios, *viz.* the ratio of $11\frac{1}{2}$ to $10\frac{1}{2}$, and the ratio of $10\frac{1}{2}$ to $9\frac{1}{2}$: now of these two ratios the former, to wit, that of $11\frac{1}{2}$ to $10\frac{1}{2}$, is somewhat less than the ratio of 11 to 10, by the observation at the beginning of this article; and the latter, to wit, that of $10\frac{1}{2}$ to $9\frac{1}{2}$, is somewhat greater than the ratio of 11 to 10, and the excess in this case is nearly equal to the defect in the former; therefore the sum of both these ratios put together, that is, the ratio of $11\frac{1}{2}$ to $9\frac{1}{2}$ will be very nearly equal to twice the ratio of 11 to 10.

2dly. As the difference between 10 and 11 is 1, add 1 to 11 and subtract it from 10, and you will have the numbers 12 and 9 whose difference is 3: I say now that 12 will be to 9, or 4 to 3, in a triplicate ratio of 11 to 10 nearly. For the ratio of 12 to 9 is resolvable into these three ratios, to wit, the ratio of 12 to 11, the ratio of 11 to 10, and the ratio of 10 to 9; and of these three ratios, the first, to wit, that of 12 to 11, is somewhat less than the middle ratio of 11 to 10, and the last, to wit, that of 10 to 9, is about as much greater; therefore the first and last ratios put together will make about twice the middle ratio of 11 to 10; therefore all these three ratios put together, to wit, the ratio of 12 to 9, will make three times the ratio of 11 to 10 nearly.

3dly.

3dly. And if increasing the difference increases the ratio proportionably, then diminishing the difference ought to diminish the ratio proportionably, that is, if the difference be reduced to half, or a third part of what it was at first, the ratio ought to be so reduced: now as the difference between 10 and 11 is 1, add $\frac{1}{4}$ to 10 and subtract it from 11, and you will have the numbers $10\frac{1}{4}$ and $10\frac{3}{4}$ whose difference is $\frac{1}{2}$, and $10\frac{1}{4}$ will be to $10\frac{3}{4}$ in a subduplicate ratio of 10 to 11 nearly; but if $\frac{1}{3}$ be added to 10 and subtracted from 11, you will then have the numbers $10\frac{1}{3}$ and $10\frac{2}{3}$ whose difference is $\frac{1}{3}$; and $10\frac{1}{3}$ will be to $10\frac{2}{3}$ in a subtriplicate ratio of 10 to 11 nearly.

Let us now try how near the ratios here found approach to the truth. By the last article the duplicate ratio of 10 to 11 is the ratio of 100 to 121, or of 1 to 1.2100; and according to the foregoing theorem it is the ratio of $9\frac{1}{2}$ to $11\frac{1}{2}$, or of 19 to 23, or of 1 to 1.2105.

By the last article the triplicate ratio of 10 to 11 is the ratio of 1000 to 1331, or of 1 to 1.331; and according to the foregoing theorem it is the ratio of 9 to 12, or of 3 to 4, or of 1.333.

By the last article the subduplicate ratio of 10 to 11 is the ratio of 1 to the square root of $\frac{11}{10}$, or of 1 to 1.04881; and according to the foregoing theorem it is the ratio of $10\frac{1}{4}$ to $10\frac{3}{4}$, or of 41 to 43, or of 1 to 1.04878.

By the last article the subtriplicate ratio of 10 to 11 is the ratio of 1 to the cube root of $\frac{11}{10}$, that is, of 1 to 1.03228; and according to the foregoing theorem, it is the ratio of $10\frac{1}{3}$ to $10\frac{2}{3}$, or of 31 to 32, or of 1 to 1.03226.

By these instances we see how near these ratios come up to the truth, even when the difference is no less than a tenth or an eleventh part of the whole: but if we suppose the difference to be the hundredth or the thousandth part of the whole, they will be much more accurate; insomuch that to multiply or divide the ratio, it will be sufficient to increase or diminish one of the numbers only. Thus 100 is to 102 in a duplicate, and to 103 in a triplicate ratio of 100 to 101; and 100 is to $100+\frac{1}{2}$ in a subduplicate, and to $100+\frac{1}{3}$ in a subtriplicate ratio of 100 to 101 nearly: and universally, *If* $A+z$ *and* $A+y$ *be any two quantities approaching infinitely near to the quantity* A, *the ratio of* $A+z$ *to* A *will be to the ratio of* $A+y$ *to* A *as the infinitely small difference* z *is to the infinitely small difference* y.

I shall draw only one example out of an infinite number that might be produced to shew the use of the foregoing proposition. Suppose then I have a clock that gains one minute every day; how much must I lengthen the pendulum to set it right? Let l be the present length of the pendulum, let x be the increment to be added to it's length in order to correct

correct it's motion, and let n be the number of minutes in one day; then it is plain that the pendulum l performs the same number of vibrations in the time $n-1$ that the pendulum $l+x$ is to perform in the time n. Now Monsieur *Huygens* has demonstrated that the times wherein different pendulums perform the same number of vibrations are in a subduplicate ratio of the lengths of those pendulums; therefore $n-1$ must be to n in a subduplicate ratio of l to $l+x$, or (which comes to the same thing) l must be to $l+x$ in a duplicate ratio of $n-1$ to n: but by the foregoing proposition, the duplicate ratio of $n-1$ to n is the ratio of $n-\frac{3}{2}$ to $n+\frac{1}{2}$, or of $2n-3$ to $2n+1$; therefore l is to $l+x$ as $2n-3$ is to $2n+1$, that is, the pendulum must be lengthened in the proportion of $2n-3$ to $2n+1$: but n the number of minutes in one day is 1440; and therefore $2n-3$ is to $2n+1$ as 2877 is to 2881, or as 719 to 720 very near; therefore the pendulum must be lengthened in the proportion of 719 to 720. *Q. E. I.*

Had the duplicate ratio of $n-1$ to n been taken only by diminishing $n-1$ to $n-2$, without meddling with the other number n, the conclusion would still have been the same; for then l would have been to $l+x$ as $n-2$ to n, as 1438 to 1440, as 719 to 720.

Having now delivered what I intended concerning the composition and resolution of ratios, it remains that I say something further concerning the application of this doctrine, and then I shall make an end of the subject.

DEFINITION 4.

299. *If two variable quantities* Q *and* R *be of such a nature, that* R *cannot be increased or diminished in any proportion, but* Q *must necessarily be increased or diminished in the same proportion; as if* R *cannot be changed to any other value* r, *but* Q *must also be changed to some other value* q, *and so changed that* Q *shall always be to* q *in the same proportion as* R *to* r; *then is* Q *said to be as* R *directly, or simply as* R. Thus is the circumference of a circle said to be as the diameter; because the diameter cannot be increased or diminished in any proportion, but the circumference must necessarily be increased or diminished in the same proportion. Thus is the weight of a body said to be as the quantity of matter it contains, or proportionable to the quantity of matter; because the quantity of matter cannot be increased or diminished in any proportion, but the weight must be increased or diminished in the same proportion.

COROLLARY 1.

If Q *be as* R *directly, then* e converso, R *must necessarily be as* Q *directly.* For let *Q* be changed to any other value q, and at the same time let R be changed to r; then since *Q* is as R, *Q* will be to q as R to r;

but

but if Q is to q as R is to r, then *vice versa*, R will be to r as Q to q: since then Q cannot be changed to q, but R must be changed to r, and that in the same proportion, it follows by this definition that R is as Q directly.

COROLLARY 2.

If Q *be directly as* R, *and* R *be directly as* S, *then will* Q *be directly as* S. For let S be changed to s, and at the same time R to r, and Q to q; then since by the supposition R is as S, R must be to r as S to s; and since again Q is as R, Q will be to q as R to r: since then Q is to q as R to r, and R is to r as S to s, it follows that Q will be to q as S to s, and consequently that Q will be as S.

COROLLARY 3.

If Q *be as* R, *and* R *be as* S; *I say then that* Q *will be as* $R \pm S$, *and also as the square root of the product* RS. For changing Q, R, S into q, r, s, since R is as S, we shall have R to r as S to s; whence by the twelfth and nineteenth of the fifth book of the Elements R will be to r as $R \pm S$ is to $r \pm s$; but Q is to q as R is to r *ex hypothesi*; therefore Q is to q as $R \pm S$ is to $r \pm s$; therefore by this definition Q will be as $R \pm S$. Again, since R is as S, R^2 will be as RS, and R as $\sqrt{RS}$; but Q is as R; therefore by the last corollary Q will be as $\sqrt{RS}$.

COROLLARY 4.

If any variable quantity as Q *be multiplied by any given number as* 5; *I say then that* 5Q *will be as* Q. For it will be impossible for Q to be increased or diminished in any proportion, but $5Q$ must be increased or diminished in the same proportion: if Q in any one case be double of Q in another, then $5Q$ in the former case must be double of $5Q$ in the latter, and so on; therefore $5Q$ is as Q.

COROLLARY 5.

If Q *be as* R, *then* Q^2 *will be as* R^2, Q^3 *as* R^3, $\sqrt{Q}$ *as* $\sqrt{R}$, &c. For let R^2 be changed in the proportion of D to E; then will R be changed in the proportion of $\sqrt{D}$ to $\sqrt{E}$; but Q is as R; therefore Q will also be changed in the proportion of $\sqrt{D}$ to $\sqrt{E}$; therefore Q^2 will be changed in the proportion of D to E: since then R^2 cannot be changed in any proportion, suppose of D to E, but Q^2 must necessarily be changed in the same proportion, it follows from this definition that Q^2 is as R^2: and the reasoning is the same in all other cases.

COROL-

COROLLARY 6.

If Q, R *and* S *be three variable quantities, and* Q *be as the product or rectangle* RS; *I say then, that* $\frac{Q}{R}$ *will always be as* S, *and* $\frac{Q}{S}$ *as* R, *and that* $\frac{Q}{RS}$ *will be a given quantity, or (which is chiefly meant by that phrase) that the quantity* $\frac{Q}{RS}$ *will always be the same, be the values of* Q, R *and* S *what they will.* For since Q is as RS, Q cannot be increased or diminished in any proportion, but RS must be increased or diminished in the same proportion; therefore $\frac{Q}{R}$ cannot be increased or diminished in any proportion, but $\frac{RS}{R}$ or S must be increased or diminished in the same proportion; therefore S is as $\frac{Q}{R}$, and $\frac{Q}{R}$ as S: and by a like proof, R will be as $\frac{Q}{S}$, and $\frac{Q}{S}$ will be as R: but if $\frac{Q}{S}$ be as R, then dividing both sides by R, we shall have $\frac{Q}{RS}$ as 1; but 1 is a quantity that neither increases nor diminishes, but is always the same; therefore the quantity $\frac{Q}{RS}$ will always be the same: and for the same reason, *If* Q *be as any single quantity, suppose* R, $\frac{Q}{R}$ *will always be the same, let* Q *and* R *be what they will.*

COROLLARY 7.

If there be four variable quantities A, B, C, D, *all in numbers, whereof* A *is as* B, *and* C *is as* D; *I say then that the product* AC *will be as the product* BD. For since A is as B, AC will be as BC, and since C is as D, BC will be as BD; therefore by the second corollary AC will be as BD; that is, AC in one case will be to AC in any other as BD in the former case is to BD in the latter.

DEFINITION 5.

300. *If two variable quantities* Q *and* R *be of such a nature, that* R *cannot be increased in any proportion whatever, but* Q *must necessarily be diminished in a contrary proportion, or that* R *cannot be diminished in any proportion whatever, but* Q *must necessarily be increased in a contrary proportion;*

portion; in a word, if R *cannot be changed in the proportion of* D *to* E, *but* Q *must necessarily be changed in the proportion of* E *to* D; *then is* Q *said to be as* R *inversely or reciprocally.* Thus if a spherical body be viewed at any considerable distance, the apparent diameter is said to be reciprocally as the distance, because the greater the distance is, the less will be the apparent diameter, and *vice versa.* Thus if a globe be supposed to move uniformly about it's *axis*, the periodical time of this motion is said to be reciprocally as the velocity with which the globe circulates; (for the quicker the circulation is, the sooner it will be over;) which is as much as to say, that the greater the velocity is with which the globe circulates, the less will be the periodical time of one revolution, and *vice versa.* Thus if the numerator of a fraction continues always the same whilst the denominator is supposed to vary, that fraction is said to be reciprocally as it's denominator, because the greater the denominator is, the less will be the value of the fraction, and *vice versa.*

COROLLARY 1.

If Q *be reciprocally as* R, *then* e converso R *will be reciprocally as* Q. For let Q be changed in the proportion of D to E, and at the same time let R be changed in the proportion of A to B; then since Q is reciprocally as R, Q must be changed in the proportion of B to A; but Q was changed in the proportion of D to E; therefore B must be to A as D to E; therefore inversely, A must be to B as E to D; but R was changed in the proportion of A to B by the supposition; therefore R was changed in the proportion of E to D. Since then Q cannot be changed in any proportion, suppose of D to E, but R must necessarily be changed in the contrary proportion of E to D, it follows from this definition that R must be reciprocally as Q.

COROLLARY 2.

If Q *be directly as* R, *and* R *be reciprocally as* S, *then* Q *must be reciprocally as* S. For let S be changed in the proportion of D to E; then since R is reciprocally as S, R must be changed in the proportion of E to D; but Q is directly as R by the supposition; therefore Q must also be changed in the proportion of E to D. Since then S cannot be changed in the proportion of D to E, but Q must necessarily be changed in the proportion of E to D, it follows from this definition that Q is reciprocally as S.

COROLLARY 3.

By a like way of reasoning, if Q *be reciprocally as* R, *and* R *be reciprocally as* S, Q *will be directly as* S.

COROL-

COROLLARY 4.

If two variable quantities Q and R be of such a nature that their product or rectangle QR is always the same; I say then that Q will be reciprocally as R. For since QR is always the same, it will be as the number 1 which neither increases nor diminishes; but if QR be as one, then Q will be as the fraction $\frac{1}{R}$ by the sixth corollary to the fourth definition. Since then Q is directly as the fraction $\frac{1}{R}$, and the fraction $\frac{1}{R}$ is reciprocally as it's denominator R by this definition, it follows from the second corollary that Q will be reciprocally as R.

COROLLARY. 5.

Every fraction is reciprocally as the same fraction inverted. Thus the fraction $\frac{R}{S}$ is reciprocally as the fraction $\frac{S}{R}$. This is evident from the last corollary; for if the fractions $\frac{R}{S}$ and $\frac{S}{R}$ be multiplied together, their product will always be unity, let R and S be what they will.

COROLLARY 6.

If Q *be reciprocally as* R, *or reciprocally as* $\frac{\mathrm{R}}{1}$, *then* Q *will be directly as* $\frac{1}{\mathrm{R}}$. For since Q is reciprocally as $\frac{R}{1}$, and $\frac{R}{1}$ is reciprocally as $\frac{1}{R}$ by the last corollary, it follows from the third corollary that Q will be directly as $\frac{1}{R}$. For the same reason, *If* Q *be reciprocally as* $\frac{1}{\mathrm{R}}$, *it will be directly as* R.

DEFINITION 6.

301. *If any quantity as* Q *depends upon several others as* R, S, T, V, X, *all independent of one another, so that any one of them may be changed singly without affecting the rest; and if none of the quantities* R, S, T *can be changed singly, but* Q *must be changed in the same proportion, nor any of the quantities* V, X, *but* Q *must be changed in a contrary proportion; then is* Q *said to be as* R *and* S *and* T *directly, and as* V *and* X *reciprocally or inversely.* Thus the fraction $\frac{RST}{VX}$ is said to be as R and S and T di-

 rectly,

rectly, and as *V* and *X* inversely, because none of the factors belonging to the numerator can be changed, but the value of the fraction must be changed in the same proportion, and none of the factors belonging to the denominator can be changed, but the value of the fraction must be changed in a contrary proportion.

N. B. *If* Q *be as* R *and* S *and* T *directly, without any reciprocals, then it is said to be as* R *and* S *and* T conjunctim, *jointly.*

A THEOREM.

302. *If* Q *be as* R *and* S *and* T *directly, and as* V *and* X *reciprocally; and if the quantities* R, S, T, V, X *be changed into* r, s, t, v, x, *and so* Q *into* q; *I say then that the ratio of* Q *to* q *will be equal to the excess of all the direct ratios taken together above all the reciprocal ones taken together: as if the ratios of* R *to* r, *of* S *to* s, *and of* T *to* t *(which I call direct ratios) when added together make the ratio of* A *to* B; *and if the ratios of* V *to* v, *and of* X *to* x *(which I call reciprocal ratios) when added together make the ratio of* C *to* D; *I say then that the ratio of* Q *to* q *will be equal to the excess of the ratio of* A *to* B *above the ratio of* C *to* D.

For supposing all but *R* to continue the same, let *R* be changed into *r*; then will *Q* be changed from it's first value in the ratio of *R* to *r* by the *hypothesis*: let now *r*, *T*, *V*, *X* continue, and let *S* be changed into *s*; then will *Q* be changed from it's last value in the ratio of *S* to *s*: in like manner if *T* be changed into *t*, *cæteris paribus*, *Q* will be changed from it's last value in the ratio of *T* to *t*: therefore if *R*, *S*, *T* be changed into *r*, *s*, *t*, *Q* will be changed from one value to another in a ratio compounded of all the direct ratios of *R* to *r*, of *S* to *s*, and of *T* to *t*; that is, *Q* will be changed in the ratio of *A* to *B*. This being so, let us now imagine *V* to be changed, *cæteris paribus*, into *v*; then will *Q* be further changed in the ratio of *v* to *V*; and if after this we imagine *X* to be changed into *x*, *Q* will be changed in the proportion of *x* to *X*, and will now be arrived at it's last value *q*: therefore if to the ratio of *A* to *B* you add the ratios of *v* to *V* and of *x* to *X*, you will have the ratio of *Q* to *q*: but to add the ratio of *v* to *V* is the same thing as to subtract the ratio of *V* to *v* by art. 296; and so again, to add the ratio of *x* to *X* is the same as to subtract the ratio of *X* to *x*; therefore if from the ratio of *A* to *B* you subtract the ratios of *V* to *v* and of *X* to *x*, you will have the ratio of *Q* to *q*; but the ratios of *V* to *v* and of *X* to *x*, when added together, make the ratio of *C* to *D ex hypothesi*; therefore if from the ratio of *A* to *B* you subtract the ratio of *C* to *D*, there will remain the ratio of *Q* to *q*; therefore the ratio of *Q* to *q* is the excess of the ratio of *A* to *B* above the ratio of *C* to *D*; or

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(which is the same thing) Q is to q in a ratio compounded of the ratio of A to B directly, and of the ratio of C to D inversely. See art. 296.

This is upon a supposition that the quantities R, S, T, V, X were changed into r, s, t, v, x one after another in time: but since the ratio of Q to q does not depend upon the intervals of time between the several changes, but will be the same whether those intervals be greater or less, it follows that the ratio of Q to q will be the same as if all these changes had been made at once. *Q. E. D.*

Corollary 1.

If the quantities R, S, T, V, X, and consequently A, B, C, D be expressed by numbers, as they must be before they can be of use in any computation; then the ratio of A to B will be the ratio of RST to rst, and the ratio of C to D will be the ratio of VX to vx; and the excess of the ratio of A to B above the ratio of C to D will be the ratio of $\frac{RST}{VX}$ to $\frac{rst}{vx}$; (see the second way of subtracting ratios in art. 296;) therefore in this case, Q will be to q as the fraction $\frac{RST}{VX}$ is to the fraction $\frac{rst}{vx}$. *Since then the fraction* $\frac{\mathrm{RST}}{\mathrm{VX}}$ *cannot be changed into* $\frac{\mathrm{rst}}{\mathrm{vx}}$, *but at the same time* Q *must be changed into* q, *and so changed that* Q *will be to* q *as* $\frac{\mathrm{RST}}{\mathrm{VX}}$ *is to* $\frac{\mathrm{rst}}{\mathrm{vx}}$, *it follows from the fourth definition that* Q *will be as the fraction* $\frac{\mathrm{RST}}{\mathrm{VX}}$; *and consequently that* Q *in any one case will be to* Q *in any other as the fraction* $\frac{\mathrm{RST}}{\mathrm{VX}}$ *in the former case is to the fraction* $\frac{\mathrm{RST}}{\mathrm{VX}}$ *in the latter.*

Corollary 2.

If there be no reciprocals, then Q *will be as the product of all the direct terms, that is, as the product* RS *if there be two of them, or as the product* RST *if there be three of them,* &c.

Scholium.

In the demonstration of the foregoing proposition as well as in the sixth definition it was supposed, that the quantities R, S, T, V, X upon which Q depended, were themselves entirely independent of one another, so as that any

any of them might be changed ſingly without affecting the reſt; and in ſuch a caſe, if Q be as R and S directly, it may be concluded to be as the product RS. But this concluſion muſt not be carried further than can be juſtified by the demonſtration: for if in any caſe the quantities R and S ſhould not be independent, if neither of them can be changed whilſt the other continues the ſame, then though no change can be made either in R or S but what will make a proportionable change in Q, yet here Q muſt not be ſaid to be as the product RS. As for example, let Q be an arc of a circle ſubtending at the diſtance R an angle whoſe quantity is repreſented by S; then it is plain that neither R nor S can be changed ſingly, but Q muſt be changed proportionably; it is plain alſo that either R or S may be changed ſingly whilſt the other remains the ſame; and therefore in this caſe it is lawful to conclude that Q is as the product RS. But let us now ſuppoſe Q to be the circumference of a circle whoſe *radius* is R, and let S be the ſide of a regular polygon of any given ſort inſcribed in that circle; as for inſtance, let S be the ſide of an inſcribed ſquare: here then it is plain that neither R nor S can be changed but Q muſt be changed proportionably; and yet if we ſhould conclude in this caſe that Q is as RS, the illation would be falſe, becauſe R and S have here as much dependence upon one another as Q upon both; for every one knows that the *radius* of a circle cannot be increaſed or diminiſhed in any proportion, but the ſide of a ſquare inſcribed in that circle muſt be increaſed or diminiſhed in the ſame proportion: in this caſe it may be concluded that Q is as $R+S$, or as $R-S$, or as the ſquare root of RS by the third corollary in art. 299, but it muſt by no means be allowed that Q is as RS; for ſhould Q be as RS, ſince in this caſe S is as R, and conſequently RS as R^2, Q would be as R^2 by the ſecond corollary in art. 299, which contradicts the ſuppoſition that Q is as R.

Examples to illuſtrate the foregoing theorem, where direct ratios are only concerned.

303. Ex. 1. *If a body moves for any time with any uniform velocity through any ſpace, that ſpace will be as the time and velocity jointly.* For if we ſuppoſe the velocity to be the ſame in all caſes, but the time to differ, then the ſpace deſcribed will be greater or leſs in proportion as the time is ſo, and therefore will be as the time: on the other hand, if we ſuppoſe the time to be the ſame in all caſes, and the velocity to differ, then the ſpace deſcribed in theſe equal times will be greater or leſs as the velocity is ſo, and conſequently will be as the velocity: laſtly let us ſuppoſe both the time and velocity to vary; then the ſpace will

vary

vary upon both these accounts, and therefore will vary in a ratio equal to the ratio wherein the time varies, and the ratio wherein the velocity varies put together; that is, the space in any one case will be to the space in any other in a ratio compounded of the ratio of the time in the former case to the time in the latter, and of the velocity in the former case to the velocity in the latter. This is universal; but if we suppose the time and velocity to be expressed by numbers, we must then say that the space described is as the product of the number representing the time multiplied into the number representing the velocity, by the second corollary in the last article; or that the space described in any one case is to the space described in any other as the product of the time and velocity in the former case is to a like product in the latter.

Ex. 2. *The quantity of matter in any body depends upon two things,* viz. *it's magnitude and density, (where by density I mean the compactness or closeness of it's matter.)* For if two bodies of equal densities but of unequal magnitudes be compared, one body must have more matter than the other, or less, according as it's solid content is greater or less, that is, according as it's magnitude is greater or less; therefore in this case the quantities of matter in any two bodies thus compared will be as their magnitudes: on the other hand, if two bodies of the same magnitude but of different densities be compared, their quantities of matter will be as their densities, because the closer the parts of a body are, so much more matter will be crouded into the same space; therefore if the bodies be different both in magnitude and density, the quantity of matter in one body will be to the quantity of matter in the other in a ratio compounded of the ratio of the magnitude of one body to the magnitude of the other, and of the ratio of the density of the former body to the density of the latter; and therefore if these quantities be represented by numbers, the quantity of matter in any body will be as it's magnitude and density multiplied together. Thus if D and d be the diameters of two globes whose densities are as E to e, the quantity of matter in the former globe will be to the quantity of matter in the latter as $D^3 \times E$ is to $d^3 \times e$; for the solid contents of all globes are as the cubes of their diameters.

Ex. 3. *The* momentum, *or force, or* impetus *with which a body moves, and with which it will strike any obstacle that lies in it's way to oppose or stop it, is as the velocity of the motion and the quantity of matter in the body jointly.* For the same quantity of matter moving with different velocities will strike an obstacle with forces proportionable to the velocities: on the other hand, different quantities of matter moving with the same velocity will strike with forces proportionable to their matter; a double

double body will strike with a double force, &c; therefore in the case where the velocity is the same, the *momentum* of a body is as the quantity of matter it contains; and in the case where the quantity of matter is the same, the *momentum* is as the velocity; therefore if neither the velocity nor the matter be the same, the *momentum* will be as the matter and velocity jointly; and in numbers, as the product of the number expressing the matter multiplied into the number expressing the velocity.

Ex. 4. *If a heavy body be suspended perpendicularly upon a lever, (by which I mean an inflexible rod moving about a fixt point in the middle,) the* momentum *or efficacy of that body to turn the lever about it's center is,* cæteris paribus, *as the weight of the body and as the distance of the point of suspension from the center of the lever jointly.* For if we suppose this distance to be the same, the *momentum* of the body to turn the lever must be greater or less according as it's weight is so, from whence that *momentum* arises: on the other hand, if we suppose the weight to be always the same, but to be removed, sometimes further from, and sometimes nearer to the center, the *momentum* of the body to turn the lever will be greater or less in proportion to the distance of the point of suspension from the center of the lever, as is demonstrated in Mechanics, and may easily be tried by experience: therefore universally, the *momentum* of the body will be as this distance and the weight of the body jointly; and in numbers is as the product of the weight multiplied into the distance.

To illustrate this, I shall put the following question. Let a body weighing five pounds be suspended at the distance of six inches from the center of a lever, and let another body of seven pounds be suspended on the same side of the center at the distance of eight inches; then let a third body of nine pound weight be suspended on the other side of the center at the distance of ten inches: *Quære* whether will these bodies sustain each other *in æquilibrio* or not; and if not, on which side will the lever dip, and with what *momentum*?

To resolve this, since we are at liberty to represent any one of these *momenta* by what numbers we please, provided the rest be represented proportionably, let us represent the *momentum* of the nine pound body by the product of it's weight and distance multiplied together, that is, by 9×10 or 90; then must the other *momenta* be represented by like products, or they would not be represented by numbers proportionable to them: therefore the *momentum* of the five pound body will be 5×6 or 30, and that of the seven pound body 7×8 or 56; and therefore the sum of the *momenta* on this side the center acting the same way will be 86: whence now it plainly appears that the lever will dip on the side of the nine pound body, because 90, the *momentum* on that side, is greater than 86,

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the sum of the *momenta* on the other side.; and since the excess of 90 above 86 is 4, it follows that 4 will be the difference of the *momenta* on one side and the other; insomuch that if any one sustains this lever immoveable, he will sustain the same force as if all the weights now upon the lever were taken away, and a single pound weight was suspended at the distance of four inches from the center of the lever: therefore when all the weights were upon the lever, if a single pound weight had been suspended at four inches distance, and on the same side of the center with the other two bodies whose weights were five and seven pounds, the whole system would then have consisted *in æquilibrio*.

Upon this theorem, that the force of a body upon a lever is as it's weight and distance from the center multiplied together, is founded the method of finding the centers of gravity of bodies, or the center of gravity of any system of bodies, let their places or positions be what they will: but I must not carry this matter any further.

Ex. 5. *If a globe be made to move uniformly in an uniform fluid, the resistance it will meet with in any given time by impinging against the particles of the fluid, will be as the density of the fluid, and as the square of the diameter of the globe, and as the square of the velocity it moves with jointly.*

To determine rightly in this case, we must here do what we all along have done, and what we always must do in like cases; that is, we must take the whole to pieces, examine every particular circumstance by itself, *cæteris paribus*, and then put them all together. First then let us suppose the same globe to move with the same velocity, but sometimes in a denser fluid, and sometimes in a rarer; then it is plain that the denser the fluid is, the more particles of it the body will be likely to meet with in any given time, and consequently the greater resistance it will suffer from them; therefore the resistance of the body, *cæteris paribus*, will be as the density of the fluid. In the next place let us suppose different globes to move in the same fluid, and with the same velocity; then since the resistance of these globes arises only from their surfaces, or rather from half their surfaces, and since the surfaces of all globes are as the squares of their diameters, it follows that the resistance these globes meet with will be as the squares of their diameters. Lastly let us suppose the same globe to move in the same fluid with different velocities; then it is plain that a globe which moves with a double velocity will strike twice as many particles of the fluid in any given time, as it would if it was to move with a single velocity: but if the body strikes twice as many particles, then twice as many particles will strike it, whence arises the resistance; therefore the resistance of a body moving with a double velocity is upon this account double of what it would have been in the

case of a single velocity: but this is not all; for it will not only strike twice as many particles, but it will strike every particle with twice the force in this case of what it would in the case of a single velocity; and therefore, since action and reaction are always equal, and since it is the reaction of the medium that creates the resistance, it follows that a body moving with a double velocity meets with four times the resistance of what it would meet with when moving with a single velocity. In like manner a body that moves with a triple velocity will act three times as strong upon three times the number of particles, and therefore will suffer nine times the resistance of what it would suffer with a single velocity; therefore the same globe moving in the same medium with different velocities will meet with a resistance proportionable to the square of the velocity it moves with. Put now all these considerations together, and the resistance of a globe moving uniformly in an uniform fluid (I mean that resistance which arises from the globe's impinging against the particles of the medium) will be as the density of the medium, as the square of the diameter of the globe, and as the square of the velocity it moves with jointly. Thus if two globes whose diameters are D and d move with velocities which are to one another as V to v in two fluids whose densities are as E to e, the resistance of the former will be to the resistance of the latter as $V^2 \times D^2 \times E$ is to $v^2 \times d^2 \times e$.

Other examples, wherein direct and reciprocal ratios are mixt together.

304. Ex. 6. *If a body be put into motion by any force directly applied, whether this force be a single impulse acting at once, or whether it be divided into several impulses acting successively; I say that the last velocity of this motion will be as the moving force directly, and as the quantity of matter in motion reciprocally.* For if different forces be applied to the same quantity of matter, the greater the force is, the greater will be the velocity, and *vice versa*; therefore in this case the velocity will be as the *vis motrix*: but if we suppose the same force to be applied to different quantities of matter, then the greater the quantity of matter is, the less will be the velocity, and *vice versa*, which I thus demonstrate. Suppose the moving force M, when applied to a certain quantity of matter as Q, will produce the velocity V; I say then that the same force M, applied to a quantity of matter equal to $2Q$, will only produce a velocity equal to $\frac{1}{2}V$: for M acting upon $2Q$ will produce the same velocity as $\frac{1}{2}M$ acting upon $1Q$; but $\frac{1}{2}M$ acting upon Q will produce a velocity equal to $\frac{1}{2}V$, because by the supposition M acting upon Q will produce the velocity V; therefore M acting upon $2Q$ will produce a velocity

locity equal to $\frac{1}{2}V$; and for the ſame reaſon, M acting upon $3Q$ will produce a velocity equal to $\frac{1}{3}V$, &c; therefore if the *vis motrix* be the ſame, the velocity of the motion produced will be reciprocally as the quantity of matter: therefore univerſally, the velocity will be as the *vis motrix* directly, and as the quantity of matter inverſely. As if M be changed into m, Q into q, and ſo V into v, the ratio of V to v will be equal to the exceſs of the ratio of M to m above the ratio of Q to q. In numbers thus; V will be to v as $\frac{M}{Q}$ is to $\frac{m}{q}$; ſee the firſt corollary in art. 302. Otherwiſe thus; the *momentum* or *impetus* with which a body moves, is the force with which it will ſtrike an object that lies in it's way to ſtop it; therefore, ſince action and reaction are equal, the force neceſſary to deſtroy any motion muſt be equal to the *momentum* with which the body moves: but the force neceſſary to deſtroy any motion is equal to the force that produced it, which we call the *vis motrix*; therefore in all motion whatever, the *vis motrix* muſt be equal to the *momentum*, and muſt be as the quantity of matter in the body moved multiplied into the velocity of the motion, becauſe the *momentum* is ſo; ſee the laſt article, example the 3d: therefore M will always be as $V \times Q$, and V as $\frac{M}{Q}$.

If M be as Q, then $\frac{M}{Q}$ will be a ſtanding quantity, and therefore the velocity V in this caſe will always be the ſame. Thus, if the weights of all bodies be proportionable to the quantities of matter they contain, they will be equally accelerated in equal times; and *vice verſa*, if all bodies, how different ſoever in their kinds and quantities of matter be equally accelerated in equal times, (as by undoubted experiments upon pendulums we find they are, ſetting aſide the reſiſtance of the air,) it follows that the weights of bodies are proportionable to their quantities of matter only, without depending upon their forms, conſtitutions, or any thing elſe.

Ex. 7. *The velocity of a planet moving uniformly in a circle round the Sun is as it's diſtance from the center of the Sun directly, and as it's periodical time inverſely.* For if two planets at different diſtances from the Sun perform their revolutions in the ſame time, that planet muſt move with the greateſt velocity that has the greateſt circumference to deſcribe; therefore in this caſe, where the periodical time is given or always the ſame, the velocity of the planet muſt be as the circumference of the circle to be deſcribed: but the circumference of every circle is as it's diameter or ſemidiameter; therefore if the periodical time be given, the velocity of a planet muſt be as it's diſtance from the Sun directly. Let us now ſuppoſe two planets revolving at the ſame diſtance from the Sun,

Sun, but in different periodical times; then it is plain that the swifter planet will perform it's revolution in less time, and *vice versa*; and therefore if the distance be given, the velocity will be reciprocally as the periodical time. Put both these cases together, and the velocity of a planet moving uniformly round the Sun will be as it's distance from the center of the Sun directly, and as it's periodical time inversely. Thus the Earth's distance from the Sun is to that of Jupiter as 10 to 52 nearly; and the Earth's periodical time is to that of Jupiter as 1 year to 12 years nearly, or as 1 to 12; therefore the Earth's velocity is to Jupiter's velocity as $\frac{10}{1}$ is to $\frac{52}{12}$, or as 120 to 52, or as 30 to 13.

This way of reasoning is applicable to all bodies moving uniformly in circles, let the law of their motions be what it will. But if (as that accurate Astronomer *Kepler* has demonstrated) the planetary motions be so tempered that their periodical times are in a sesquiplicate ratio of their distances, or (which is the same thing by art. 297) that the squares of their periodical times are as the cubes of their distances, we shall then have a more simple way of expressing the velocity of a planet thus: let V be the velocity and D the distance of any planet from the Sun, and let T be the periodical time; then since, from what has been said, V is as $\frac{D}{T}$, we shall have V^2 as $\frac{D^2}{T^2}$; but according to *Kepler*'s proportion, T^2 is as D^3, and $\frac{D^2}{T^2}$ as $\frac{D^2}{D^3}$ or as $\frac{1}{D}$; therefore V^2 is as $\frac{1}{D}$, and V as $\frac{1}{\sqrt{D}}$; that is, in this case, the velocity of a planet is reciprocally in a subduplicate ratio of it's distance from the Sun. So the velocity of a planet whose distance is D is to the velocity of a planet whose distance is d as $\sqrt{d}$ is to $\sqrt{D}$, or as 1 is to $\sqrt{\frac{D}{d}}$.

Ex. 8. *If a wheel turns uniformly about it's* axis, *the time of one round will be as the diameter of the wheel directly, and as the absolute velocity of every point in the circumference of the wheel inversely.* For if the circumference of a great wheel moves with the same velocity as the circumference of a small one, the periodical time of the former wheel will be as much greater in proportion than the periodical time of the latter as the circumference of the former wheel is greater than the circumference of the latter, or as the diameter of the former is greater than the diameter of the latter; therefore if the velocity of the wheel's circumference be given, the periodical time will be as the diameter of the wheel directly: let us now suppose the velocity of the circumference of the same wheel to be in any case increased; then will the periodical time be dimi-

diminiſhed in a contrary proportion, and *vice verſa*; therefore if the diameter of a wheel be given, the periodical time will be reciprocally as the velocity of the circumference; therefore if neither the diameter nor the velocity of the circumference be given, the periodical time will be as the diameter of the wheel directly, and as the abſolute velocity of every point in the circumference inverſely. In numbers the periodical time will be as $\frac{D}{V}$.

Ex. 9. *The relative gravity of any ſpecies of bodies is as the abſolute weight of any body of that ſpecies directly, and as it's magnitude inverſely*; where by the magnitude or bulk of a body is meant the quantity of ſpace it takes up, and not the quantity of matter it contains.

All bodies of the ſame kind are ſuppoſed to weigh in proportion to their magnitudes; and therefore if a body of any one kind be compared with a body of the ſame magnitude of another kind, the proportion of their weights will always be the ſame, let their common magnitude be what it will; and hence ariſes the compariſon in general of the weight of one ſpecies of bodies with the weight of another: if a cubic inch of gold be 19 times as heavy as a cubic inch of water, then a cubic foot of gold will be 19 times as heavy as a cubic foot of water, *&c*; and ſo we pronounce in general that gold is 19 times as heavy as water, though we mean bulk for bulk. In this ſenſe therefore may any one ſpecies of bodies be ſaid to be heavier or lighter than another, in proportion as any one body of the former ſpecies is heavier or lighter than a body of the ſame magnitude of the latter, which is the ſame in effect with the firſt part of my aſſertion. Let us now compare bodies of the ſame weight, but of different magnitudes; and then it will appear that the ſpecific gravities of theſe bodies, that is, of the ſeveral ſpecies to which they belong, will be reciprocally as the magnitudes of the bodies compared: thus if a cubic inch of gold be as heavy as 19 cubic inches of water, then the ſpecific gravity of gold will be to the ſpecific gravity of water, not as 1 to 19, but as 19 to 1; for if 1 cubic inch of gold be as heavy as 19 cubic inches of water, then 1 cubic inch of gold will be 19 times as heavy as 1 cubic inch of water; and therefore, from what has been ſaid in the former caſe, the ſpecific gravity of gold will be to the ſpecific gravity of water as 19 to 1. Put both theſe caſes together, and the relative gravity of any ſpecies of bodies will be as the abſolute weight of any one body of that ſpecies directly, and as it's magnitude inverſely. Thus if in numbers P and p be the weights of two globes whoſe diameters are D and d, the ſpecific gravities of the metals out of which theſe two globes were formed are as $\frac{P}{D^3}$ to $\frac{p}{d^3}$.

Ex.

EX. 10. *If a body as* A *gravitates toward the center of a planet as* B *at the distance* D; *I say then that the weight of* A *will be as the quantity of matter in* A *directly, and as the quantity of matter in* B *directly, and as the square of the distance* D *inversely.* For the weight of the whole body A towards B arises, *cæteris paribus*, from the weight of all it's parts; and therefore in such a case will be as the quantity of matter in A. Again, the weight of A towards the whole planet B arises, *cæteris paribus*, from the weight of A to all the parts of B; and therefore in such a case will be as the quantity of matter in B. Lastly, if the quantities of matter in A and B continue the same, and the distance D be supposed to vary, the great *Newton* has demonstrated that the weight of A towards B will be reciprocally as the square of the distance D. Therefore if neither the quantities of matter in A and B, nor the distance D be the same, the weight of A towards B will be as the quantity of matter in A directly, and as the quantity of matter in B directly, and as the square of the distance D inversely. Thus if A and B be numbers representing the quantities of matter in the bodies A and B respectively, the weight of A towards B at the distance D will be as $\frac{AB}{D^2}$, that is, the weight of A towards B at the distance D will be to the weight of a towards b at the distance d as the fraction $\frac{AB}{D^2}$ is to the fraction $\frac{ab}{d^2}$.

Hence the weight of A towards B will be equal to the weight of B towards A, since both will be represented by the same quantity $\frac{AB}{D^2}$.

Another way of treating the examples in the two last articles.

305. *If there be ever so many quantities, and these all heterogeneous to one another, we are at liberty to represent them by what numbers we please, or even all by unity itself, provided we take care to represent all other quantities of like kinds by proportionable numbers.* Thus I am at liberty to call any quantity of time I please 1, or any degree of velocity 1, or any quantity of space 1; but then I must take care to call a double time, or a double velocity, or a double space by the number 2, and so on. This consideration suggests to us another way of treating the examples in the two last articles, somewhat different from the former; which, as it may be explained by a bare instance or two, I shall give the learner as follows.

In

In the first example we were taught that the ſpace deſcribed by a body moving uniformly for any time, and with any velocity, is in numbers as the time and velocity multiplied together; which may alſo be demonſtrated thus: ſuppoſe that a body moving uniformly in ſome known time called 1, and with ſome velocity called 1, ſhall deſcribe a ſpace which we will alſo call 1; then if in the time 1, and with the velocity 1, there be deſcribed the ſpace 1, it is plain that in the time T, and with the velocity 1, there will be deſcribed the ſpace T; but if in the time T, and with the velocity 1, there be deſcribed the ſpace T, then in the time T, and with the velocity V there will be deſcribed the ſpace VT, and that, let the quantities V and T be what they will; and therefore in all caſes, the ſpace will be as $T \times V$.

Again, in the ſixth example it was ſhewn that if any moving force as M be directly applied to any body whoſe quantity of matter is Q, the velocity thereby produced will be as $\frac{M}{Q}$: for a further demonſtration whereof, let us ſuppoſe that ſome known force called 1, when applied to ſome quantity of matter called 1, will produce the velocity 1; then will the force 2 applied to the ſame quantity of matter 1 produce the velocity 2; but if the force 2 when applied to the quantity of matter 1 produces the velocity 2, then the ſame force 2 applied to a quantity of matter as 3 will produce a velocity equal to a third part of the former, to wit $\frac{2}{3}$; and for the ſame reaſon the force M applied to a quantity of matter as Q will produce the velocity $\frac{M}{Q}$; and therefore this velocity will always be as $\frac{M}{Q}$.

It is not impoſſible but that ſome of my leſs judicious readers may be inclined to think I have ſpun out this ſubject to too great a length: but I eaſily perſwade my ſelf that there are none who have thoroughly conſidered the very great uſefulneſs and importance of this doctrine, eſpecially in Mechanical and Natural Philoſophy, but will readily acquit me of this charge; and the more ſo, becauſe none that I know of have digeſted theſe matters into a ſyſtem, or have written ſo diſtinctly upon them as the importance of the ſubject requires.

THE ELEMENTS of ALGEBRA

BOOK VIII. IN TWO PARTS.

I. The application of Algebra to plain Geometry.

II. Of Prisms, Cylinders, Pyramids, Cones and Spheres.

PART I.

Requisites for applying Algebra to Geometry: and how numeral expressions of geometrical magnitudes are to be understood.

306. HITHERTO I have contented myself with treating of Algebra as it relates to numbers only: I shall in the next place proceed to apply that art to Geometry, and shew that it is no less useful in the resolution of geometrical than arithmetical problems. Therefore it is now absolutely necessary that the learner before he enters upon this part, becomes tolerably well acquainted with the common principles of Geometry, and especially such theorems as are usually referred to in the resolution of geometrical problems; such as the Pythagoric theorem, which is the 47th proposition of the first book of *Euclid*'s elements; the doctrine of similar triangles delivered in the sixth book; the nature and properties of the circle, exhibited in the third book; the nature of proportion, and the several variations of proportionable quantities, enumerated and explained in the fifth book; and so on. It is true indeed that here in the matter of proportion, I thought I foresaw some difficulties, or discouragements at least, which if not removed, a young beginner would scarce of himself be able to get over; but these I have taken care to obviate and clear up in the foregoing book in such a manner, that it is now to be hoped the fifth book of the Elements, as it is there delivered, will be found as easy and

and as intelligible as any other part of the Elements whatever, if not more so; and therefore I shall now proceed to other matters.

In representing lines by numbers we are at liberty in any problem to represent what line we please by unity, provided that in that problem all other lines be represented by proportionable numbers. Thus if an inch be represented by an unit, a foot must be represented by the number 12; if a foot be represented by an unit, a yard must be represented by the number 3, and so on; but it is not necessary that this standard-line represented by unity should always be expressed: thus when the three sides of a triangle are represented by the numbers 3, 4 and 5, those sides may be 3, 4 and 5 inches, 3, 4 and 5 feet, or 3, 4 and 5 yards, &c, provided that all other lines to which these are to be compared be proportionably represented.

As to surfaces, if any number as 10 represents an area, that area must be looked upon as equivalent to ten equal squares whose sides are such lines as are represented by unity.

Lastly, if any number as 10 represents the content of any solid, that solid content must be looked upon as equivalent to ten equal cubes whose sides are units. Thus then if the number 1 represents a line of a foot long, the number 10, when it represents a line, will signify a line ten feet long; when it represents an area, it will signify ten square feet, and when it represents a solid, it will signify ten cubic feet.

PROBLEM I.

307. *It is required, having given* a *and* b *the two legs of a right-angled triangle, whereof* a *is the greater, to find it's hypotenuse, without the fortyseventh of the first Element.*

SOLUTION. (See Plate I. Fig. 2.)

Out of eight triangles, all similar and equal to the triangle proposed, let four right-angled parallelograms be formed, and disposed as in the scheme, to wit, AK, BL, CM and DN; then from the uniformity and constitution of the figure we shall have three squares, to wit, $ABCD$ the greatest, $EFGH$ the middlemost, and $KLMN$ the least. It is further evident that the greatest square exceeds the middlemost by four of the triangles above mentioned, and that the middlemost exceeds the least by the other four triangles; and consequently that the middlemost square is an arithmetic mean between the greatest and the least. But the side of the greatest square is $AB = AE + EB = a + b$; and the side of the least square is $KL = EL - EK = a - b$: therefore the area of the greatest square is $a^2 + 2ab + b^2$, and the area of the least

 square

ſquare is $a^2 - 2ab + b^2$, and an arithmetic mean between theſe two areas is $a^2 + b^2$; therefore the area of the middle ſquare is $a^2 + b^2$: but the middle ſquare is the ſquare of the hypotenuſe of the triangle propoſed; therefore *If* a *and* b *be the legs of any right-angled triangle, the ſquare of the hypotenuſe will be* $a^2 + b^2$, *and the hypotenuſe itſelf will be* $\sqrt{a^2 + b^2}$. Q. E. I.

Note, that by this problem the relation between the hypotenuſe and the legs of a right-angled triangle is inveſtigated: otherwiſe, the two legs being given, to find the hypotenuſe nothing more is required, than to draw a right line equal to one of the legs and perpendicular to the other at it's extremity, as *HA* perpendicular to *AE*, and to joyn *HE*.

SCHOLIUM. *(Fig. 3.)*

Though ſurds cannot be expreſſed in rational numbers, yet by the help of the foregoing propoſition they may be as exactly expreſſed by lines as any rational numbers whatever can be. *Let the line* AB *repreſent unity, and perpendicular to it draw the line* BK, *upon which ſet off* BC = AB, BD = AC, BE = AD, BF = AE, BG = AF, *and* BH = AG: *I ſay then that if* AB *expreſſes unity*, BC *will expreſs* 1 *or* $\sqrt{1}$, BD $\sqrt{2}$, BE $\sqrt{3}$, BF $\sqrt{4}$, BG $\sqrt{5}$, BH $\sqrt{6}$, &c. For firſt, $AB^2 = 1$, $BC^2 = 1$, therefore $AC^2 = 2$; but $BD = AC$ by the conſtruction; therefore $BD^2 = AC^2 = 2$; therefore $BD = \sqrt{2}$. Secondly, $AB^2 = 1$, $BD^2 = 2$, therefore AD^2 or $BE^2 = 3$; therefore $BE = \sqrt{3}$. Thirdly, $AB^2 = 1$, $BE^2 = 3$, therefore AE^2 or $BF^2 = 4$; therefore $BF = \sqrt{4}$; and ſo of the reſt. Since then the lines *BD*, *BE*, *BF*, &c may be as exactly taken as any other lines repreſenting any rational numbers whatever, it follows that rational and irrational numbers are equally expreſſible in Geometry. Could *Euclid*'s *poſtulata* be perfectly and exactly executed, the lines *BD*, *BE*, &c would expreſs their reſpective ſurds to a degree of perfect exactneſs, which is more than can be done by rational numbers: but as theſe *poſtulata* cannot be exactly executed, there muſt be ſome errors in the drawing of the lines; and in ſuch a caſe it is no wonder if theſe ſurds may be more exactly repreſented in numbers than in lines, though even here it muſt be obſerved that there are no errors incident to the repreſentation of ſurds by lines, but what the repreſentation of rational numbers by lines is equally liable to.

Note, that the ſcale here deſcribed is that referred to in article 202, ſcholium 1.

PROBLEM 2.

308. *To find the area of a triangle whoſe three ſides are given in numbers.*

INVESTI-

INVESTIGATION.

1ſt. The ſum and difference of any two numbers multiplied together will give the difference of their ſquares, and *vice verſa*: thus $\overline{a+y}\times\overline{a-y}=a^2-y^2$; and thus again $c^2-d^2=\overline{c+d}\times\overline{c-d}$. See art. 9th, Ex. 4th.

2dly. Let the triangle propoſed be ABC (*Fig.* 4,) whereof let $AB=a$, $BC=b$, and $CA=c$; and let $a+b+c$ (or the ſum of all the ſides) $=2s$; then if $2a$ be ſubtracted from both ſides, you will have $-a+b+c=2s-2a$: and for a like reaſon $a-b+c=2s-2b$, and $a+b-c=2s-2c$.

3dly. Draw the perpendicular AD, and the two right-angled triangles ADB and ADC furniſh the two following equations, $AD^2+DB^2=AB^2$, and $AD^2+DC^2=AC^2$.

4thly. Make $AD=x$, $BD=y$, and conſequently $CD=b-y$, and the two equations in the laſt ſtep, according to this notation, will ſtand thus:

$$x^2+y^2 \quad * \quad *=aa,$$

and

$$x^2+y^2-2by+b^2=cc.$$

Subtract the latter equation from the former, and you will have

$$* \quad * +2by-b^2=a^2-c^2;$$

whence $2by=a^2+b^2-c^2$.

5thly. Since by the laſt ſtep $2by=a^2+b^2-c^2$, we have $-2by=-a^2-b^2+c^2$, and $2ab-2by$ or $2b\times\overline{a-y}=-a^2+2ab-b^2+c^2$; make $a-b=d$, and you will have $a^2-2ab+b^2=d^2$, and $-a^2+2ab-b^2=-d^2$, and $-aa+2ab-b^2+c^2=c^2-d^2$; therefore $2b\times\overline{a-y}=c^2-d^2=$ (by the firſt) $\overline{c-d}\times\overline{c+d}=\overline{c-a+b}\times\overline{c+a-b}=$ (by the ſecond) $\overline{2s-2a}\times\overline{2s-2b}=4\times\overline{s-a}\times\overline{s-b}$; therefore from firſt to laſt, $2b\times\overline{a-y}=4\times\overline{s-a}\times\overline{s-b}$, and $\frac{2b}{4}\times\overline{a-y}$ or $\frac{b}{2}\times\overline{a-y}=\overline{s-a}\times\overline{s-b}$.

6thly. Again, ſince by the fourth ſtep, $2by=a^2+b^2-c^2$, we have $2ab+2by$ or $2b\times\overline{a+y}=a^2+2ab+b^2-c^2$; make $a+b=d$, and you will have $a^2+2ab+b^2=d^2$, and $a^2+2ab+b^2-c^2=d^2-c^2=\overline{d-c}\times\overline{d+c}=\overline{a+b-c}\times\overline{a+b+c}=\overline{2s-2c}\times 2s=4\times\overline{s-c}\times s$; therefore from firſt to laſt, we have $2b\times\overline{a+y}=4\times\overline{s-c}\times s$, and $\frac{b}{2}\times\overline{a+y}=\overline{s-c}\times s$.

7thly. Multiply the equations found in the two last steps together, *viz.* $\frac{b}{2}\times\overline{a-y}=\overline{s-a}\times\overline{s-b}$, and $\frac{b}{2}\times\overline{a+y}=\overline{s-c}\times s$, and you will have (by the first) $\frac{b^2}{4}\times\overline{a^2-y^2}=\overline{s-a}\times\overline{s-b}\times\overline{s-c}\times s$.

8thly. But by the fourth step $a^2=x^2+y^2$, and $a^2-y^2=x^2$; therefore $\frac{b^2x^2}{4}=\overline{s-a}\times\overline{s-b}\times\overline{s-c}\times s$.

9thly. Extract the square root of both sides and you will have $\frac{bx}{2}$, or $AD\times\frac{1}{2}BC$, or the area sought, equal to the square root of the product $\overline{s-a}\times\overline{s-b}\times\overline{s-c}\times s$. In words thus:

From half the sum of the sides subtract the three sides severally, and mark down the remainders; then if these three remainders and that half sum be multiplied together by a continual multiplication, the square root of the product will be the area sought.

Note. If the perpendicular falls without the triangle, (as in *Fig.* 5,) the investigation of this theorem will still be the same, only now the sign of y will be changed.

EXAMPLE 1.

Let the three sides of the triangle whose area is sought be 13, 14 and 15, and the sum of these sides will be 42, and half their sum 21; from this half sum 21 subtract the three sides 13, 14 and 15 severally, and the remainders will be 8, 7 and 6 respectively; multiply now these four numbers 21, 8, 7 and 6 together by a continual multiplication, and you will have $21\times8=168$, $168\times7=1176$, and $1176\times6=7056$, whose square root is 84; therefore 84 is the area sought.

Hence may a perpendicular let fall from any angle be found. For since the perpendicular multiplied into half the base gives the area, if on the contrary, the area be divided by half the base, the quotient will be the perpendicular. Thus in the foregoing example, if the side 14 be made the base, divide 84 the area by 7 half the base, and the quotient 12 will be the perpendicular.

EXAMPLE 2.

Let the three sides be 13, 4 and 15; then we shall have $\frac{13+4+15}{2}=16$, $16-15=1$, $16-13=3$, and $16-4=12$, and $1\times3\times12\times16=576$, whose square root 24 is the area sought.

EXAM-

EXAMPLE 3.

Let the three ſides be 7, 8 and 15; then we ſhall have $\frac{7+8+15}{2}=15$, $15-15=0$, $15-8=7$, and $15-7=8$, and $0\times7\times8\times15=0$, whoſe ſquare root is 0; therefore the area of this triangle is 0: the reaſon whereof is an abſurdity in the ſuppoſition; for we ſuppoſed a triangle whoſe two ſides 7 and 8 were equal to the third ſide 15, whereas in every triangle any two ſides taken together ought to be greater than the third. Had we propoſed a triangle whereof two ſides were leſs than the third, the ſuppoſition would have been ſtill more abſurd; and in this caſe the area would have been the ſquare root of a negative quantity, which is impoſſible: therefore the problem needed no limitation, ſince the canon limits itſelf.

COROLLARY.

If 1 be the ſide of an equilateral triangle, half the ſum of the ſides will be $\frac{1}{2}$, and each remainder will be $\frac{1}{2}$; but $\frac{3}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{3}{16}$; and therefore the area of the triangle will be $\frac{\sqrt{3}}{4}$, and it's perpendicular altitude $\frac{\sqrt{3}}{2}$; therefore *The baſe of an equilateral triangle is to it's perpendicular altitude as* 1 *to* $\frac{\sqrt{3}}{2}$, *as* 2 *to* $\sqrt{3}$, *or nearly as* 2000 *to* 1732, *or as* 15 *to* 13 *nearly.*

A LEMMATICAL PROBLEM.

309. *To find in rational numbers two right-angled triangles, having one leg the ſame in both.*

SOLUTION.

By art. 12 find any two right-angled triangles expreſſed in rational numbers: then if the three ſides of each triangle be multiplied into either leg of the other, you will have two right-angled triangles ſimilar to the former, which will have one leg the ſame in both.

Let the ſides of the two right-angled triangles found by art. 12 be a, b, c, and d, e, f, whereof c and f are hypotenuſes: then if the ſides a, b and c of the firſt triangle be multiplied by d, one of the legs of the latter; and if the ſides d, e and f of the latter triangle be multiplied by a, one of the legs of the former; there will ariſe other two triangles, *viz.* ad, bd, cd, and ad, ae, af. Now that theſe are right-angled triangles, and ſimilar to the former, is evident from art. 224; and that there is one leg the ſame in both, to wit ad, is evident from the operation.

If

If the two legs that are made multiplicators, as in the present case a and d, be not the least in their proportion, let a be to d as r to s, and let r and s be the least in the proportion of a to d: then if instead of multiplying the first triangle by d, and the second by a, the first had been multiplied by s, and the second by r, we should have had these two triangles, more simple than the former, to wit, as, bs, cs; dr, er, fr, wherein the leg as of one would have been equal to the leg dr of the other: for since a is to d as r to s, as and dr are the products of the extremes and means.

EXAMPLE.

Let the triangles found by art. 12 be 3, 4 and 5, and 5, 12 and 13; then if 4 and 12 be multiplicators, since 4 is to 12 as 1 to 3, multiply the first triangle by 3, and the second by 1, and you will have the triangles 9, 12 and 15, and 5, 12 and 13, having one leg 12 the same in both.

PROBLEM 3.

310. *To find as many oblique-angled triangles as we please whose sides and areas are all expressible by rational numbers.*

SOLUTION. *(Fig. 4, 5.)*

Find by the last article two right-angled triangles ADB and ADC having one leg AD in common: then if by means of this common leg, the triangle ADB be added to the triangle ADC, as in the former scheme *(Fig. 4,)* or subtracted from it, as in the latter *(Fig. 5,)* we shall have in both cases a triangle as ABC whose sides and area are all expressed in rational numbers. For first, the sides AB and AC will be the hypotenuses of the constituent right-angled triangles; and secondly, the base BC will be either the sum or difference of the bases of those triangles; and lastly the area, which is half the product of BC multiplied into AD, must be expressed in rational numbers, because the factors BC and AD are so.

EXAMPLE.

Let the right-angled triangles found by the last article be 12, 5 and 13, and 12, 9 and 15, whose hypotenuses are 13 and 15: then, since the sum of the bases 9 and 5 is 14, and their difference 4, we shall have two oblique-angled triangles, whose sides and areas are all in rational numbers; to wit, the acute-angled triangle 13, 14 and 15, and the obtuse-angled triangle 13, 4 and 15, as in the two first examples in the last article but one.

PROBLEM 4.

311. *Having given the three sides of any triangle in numbers, to find the semidiameter of an inscribed circle.*

SOLU-

SOLUTION. (*Fig.* 6.)

Let ABC be the triangle whose three sides are given, and let DEF be the inscribed circle touching the three sides AB, BC, CA in the three points D, E, F respectively. From G the center of the circle draw the lines GA, GB, GC, GD, GE, GF, and the three last lines GD, GE, GF will be perpendiculars to the three tangent sides, by the 18th proposition of the third book of the Elements. By this means the triangle ABC is divided into the lesser triangles GAB, GBC, GCA, having equal perpendicular altitudes GD, GE, GF: the area of the triangle GAB will be $\frac{1}{2}AB\times GD$; the area of the triangle GBC will be $\frac{1}{2}BC\times GE$ or GD; and the area of the triangle GCA will be $\frac{1}{2}CA\times GF$ or GD; and all these areas put together will be $\overline{\tfrac{1}{2}AB+\tfrac{1}{2}BC+\tfrac{1}{2}CA}\times GD$; or if we put s for the sum of all the three sides of the great triangle ABC, the areas of the three lesser ones added into one sum will be $GD\times\frac{1}{2}s$; but these areas put together constitute the area of the great triangle ABC: *Find therefore by art.* 308 *the area of the triangle* ABC, *and call it* q, *and you will have* $GD\times\frac{1}{2}s=q$, *and* $GD=\frac{2q}{s}$: *if therefore twice the area of any triangle be divided by the sum of the sides, the quotient will be the semidiameter of an inscribed circle.* Q. E. I.

COROLLARY 1.

Since twice the area of any triangle may be found by multiplying the base into the perpendicular altitude, it follows, that *as the sum of all the three sides is to the base alone, so is the perpendicular altitude to the* radius *of an inscribed circle.*

COROLLARY 2.

Hence in an equilateral triangle, the radius *of an inscribed circle is one third of the perpendicular altitude.* And if the side be called a, and consequently the perpendicular altitude $\frac{1}{2}a\times\sqrt{3}$ by the corollary in art. 308, we shall have, in an equilateral triangle, $GD=\frac{1}{2}a\times\frac{\sqrt{3}}{3}$, and $GD^2=\frac{1}{4}aa\times\frac{3}{9}=\frac{1}{12}aa$, and $GD=\sqrt{\frac{aa}{12}}$: therefore *If the square of the side of an equilateral triangle be divided by* 12, *the square root of the quotient will be the* radius *of an inscribed circle.*

COROLLARY 3

From the demonstration of this fourth problem it plainly appears, that *The area of any polygon circumscribed about a circle is equal to the* radius *multi-*

multiplied into half the perimeter of the polygon; and consequently that the area of such a polygon is equal to that of a triangle, whose base is the perimeter of the polygon, and whose perpendicular altitude is the radius *of the inscribed circle.*

COROLLARY 4.

Since the properties described in the last corollary do not at all depend on the number of sides of the polygon, it follows that these properties must subsist even when the number of sides is infinite: but a polygon of an infinite number of sides circumscribed about a circle differs nothing from the circle itself; therefore *Every circle is equal to a triangle whose base is a right line equal to the circumference, and whose perpendicular altitude is the* radius: *and the area of every circle is had by multiplying the* radius *into half the circumference.*

COROLLARY 5.

If an equilateral polygon be incribed in a circle, such a polygon will be equal to a triangle whose base is half it's perimeter, and whose perpendicular altitude is equal to a line drawn from the center of the circle perpendicular to any one side of the polygon: for they are all equal, the polygon being supposed equilateral.

PROBLEM 5.

312. *Having given the three sides of any triangle in numbers, to find the diameter of a circumscribing circle.*

SOLUTION. *(Fig, 7.)*

Let the inscribed triangle be ABC, whose sides are all given in numbers, and let AD be the diameter of the circumscribing circle; join BD, and let AE be perpendicular to BC; and you will have two similar triangles, AEC and ABD; for the angle AEC is a right one by construction, and the angle ABD is a right one as being in a semicircle, and moreover the angles C and D are equal, as insisting upon the same arc AB; therefore as AE is to AC in one triangle, so is AB to AD in the other; therefore by multiplying extremes and means we have $AE \times AD = AB \times AC$, and $AD = \frac{AB \times AC}{AE}$; multiply both numerator and denominator of this last fraction by BC, and you will have $AD = \frac{AB \times AC \times BC}{AE \times BC}$: but $AB \times AC \times BC$ is the product of all the three sides of the triangle multiplied together by a continual multiplication; and

$AE \times BC$

$AE \times BC$ is twice the area of the triangle; therefore *If the product of all the three sides of any triangle multiplied together by a continual multiplication be divided by twice the area, the quotient will be the diameter of a circumscribing circle.* Q. E. I.

COROLLARY.

If a be the side of an equilateral triangle, the product of a continual multiplication of all the sides will be a^3, and twice the area of the triangle will be $a^2 \times \frac{\sqrt{3}}{2}$ by the corollary in art. 308; therefore the product divided by this double area will be $\frac{2a}{\sqrt{3}}$; therefore the diameter of a circumscribing circle will be $\frac{2a}{\sqrt{3}}$, and the *radius* $\frac{a}{\sqrt{3}}$, whose square is $\frac{aa}{3}$: therefore *If the square of the side of an equilateral triangle be divided by* 3, *the square root of the quotient will be the* radius *of a circle circumscribed about it.*

Whence the radius *of a circle circumscribed about an equilateral triangle will be double the* radius *of a circle inscribed:* for the square of the *radius* of the former circle will be to the square of the *radius* of the latter as $\frac{a^2}{3}$ is to $\frac{a^2}{12}$ by the 2d corollary in art. 311, that is, as 4 to 1; therefore the *radii* themselves will be as 2 to 1.

A LEMMA.

313. *If in a triangle a line be drawn from any angle to the opposite base, it will divide all lines parallel to the base in the same proportion as it divides the base itself.* (*Fig.* 8.)

From the angle A of the triangle ABC draw the line AD cutting the base BC in D, and the line EG parallel to the base in F: I say then that EF will be to FG as BD to DC. For by similar triangles, EF will be to BD as AF to AD; but AF is to AD as FG is to DC: therefore by the 11th of the fifth book of *Euclid*, EF is to BD as FG to DC, and *permutando*, EF is to FG as BD is to DC. *Q. E. D.*

PROBLEM 6.

314. *To find the center of gravity of any given triangle; that is, to find such a point in the plane of a given triangle as being sustained upon the point of a pin, or otherwise, the whole figure shall be* in æquilibrio.

SOLUTION. *(Fig. 9.)*

Let the triangle be ABC, *and from any angle as* A *draw the line* AD *bisecting the opposite side in* D; *then if from the point* A *towards* D *be set off* AE *equal to two thirds of the line* AD, *the point* E *will be the center of gravity required.*

For first, as the line *AD* bisects the base *BC*, it will bisect all other lines parallel to it, by the lemma foregoing: therefore if we imagine this whole figure to be made up of an infinite number of infinitely small physical lines or wires, all parallel to the base *BC*, whereof the greatest is the base *BC*, and all the rest shorten by degrees till they vanish at *A*; if, I say, we imagine this figure to be so constituted, the line *AD* will pass through the middle of all these wires, and therefore all the elementary wires will be *in æquilibrio* with respect to the line *AD*; therefore the center of gravity of the figure must be somewhere in that line *AD*. In like manner, if from the angle *B* the line *BF* be drawn, bisecting the opposite side *AC* in *F*, the center of gravity must be somewhere in the line *BF*; therefore if the point *E* be the intersection of the two lines *AD* and *BF*, the center of gravity must be in the point *E*; for otherwise it could not be in both those lines. Join *FD*; and because the line *FD* cuts the sides *AC* and *BC* proportionably, it must be parallel to *AB*; and so the triangles *FCD* and *ACB* will be similar, as also the triangles *EDF* and *EAB*; therefore *AE* is to *ED* as *AB* is to *FD*, or as *BC* to *CD*, or as 2 to 1: since then *AE* is to *ED* as 2 to 1, it follows by composition of proportion that *AD* is to *ED* as 3 to 1; therefore *ED* is one third, and *AE* two thirds of the whole line *AD*. *Q. E. D.*

SCHOLIUM.

By the help of this proposition may the centers of gravity of all other right-lined figures be found; and by a way of reasoning not unlike to this, may also be found the center of gravity of any triangular pyramid, and consequently of any other pyramid or cone whatsoever. But I am too much out of my way already.

PROBLEM 7. *(Fig. 4, 5.)*

315. *It is required, having given the base* BC *of any triangle, together with the adjacent angles* B *and* C, *to find it's perpendicular altitude.*

N. B. This problem is easily resolved geometrically, if from *B* and *C* the two ends of the given base be drawn the lines *BA* and *CA* in the angles given, by the 23d proposition of the first Element; for then *ABC*

ABC will be the triangle propoſed, and AD will be the perpendicular ſought. But I ſhall give the arithmetical ſolution of this problem, after having firſt advertiſed the reader, that an angle as ABD is ſaid to be given, when (if a point as A be aſſumed in one of the legs, and a line as AD be drawn perpendicular to the other) the proportion of AB to BD, or of AB to AD, or of AD to DB is given; for any of theſe proportions being given, the angle B will be found in degrees and minutes by the help of trigonometrical tables; and *vice verſa*, if the angle B be given in degrees and minutes, any of the foregoing proportions will eaſily be had from the ſame tables.

SOLUTION.

Call the given baſe BC, b; let AD be to DB as r to p, and moreover let AD be to DC as r to q; then if the unknown altitude AD be called x, the ſegment BD will be $\frac{px}{r}$, and the ſegment CD $\frac{qx}{r}$, and (*Fig.* 4) the ſum of the ſegments will be $\frac{px+qx}{r}$: but the ſegments BD and CD both together make the whole baſe $BC=b$; therefore $\frac{px+qx}{r}=b$; whence AD or $x=\frac{br}{p+q}$.

Since AD is to BD as r to p, it follows that if r continues the ſame, the nearer the point D approaches to the point B, *cæteris paribus*, the leſs will be the quantity p; if D coincides with B, as is the caſe when the angle ABC is a right one, we ſhall have $p=0$; therefore when D lies on the other ſide of B in the line CB produced, (as in *fig.* 5,) that is, when the angle ABC it obtuſe, the quantity p will then be negative, and we ſhall have the perpendicular $AD=\frac{br}{q-p}$.

PROBLEM 8. (*Fig.* 10.)

316. *Let* ACDB, AEFB *and* AGHB *be three given right-angled parallelograms, all ſituate upon the ſame baſe* AB, *and all the ſame way, ſuppoſing the rectangle* AGHB *to be the greateſt, and the rectangle* ACDB *to be the leaſt: It is reqnired to draw the line* KLMN *parallel to* AG *or* BH, *and cutting the lines* GH *in* K, EF *in* L, CD *in* M, *and* AB *in* N; *ſo that the ſum of the rectangles* AK *and* BM *may be equal to the middle rectangle* AF.

SOLUTION.

Call AB p, AG q, AE r, and AC s; call alſo the unknown part AN x, and conſequently BN $p-x$, and the rectangle AK or $AG\times AN$ will be qx, and the rectangle BM or $BD\times BN$ or $AC\times BN$ will be $ps-sx$, and laſtly the rectangle AF or $AB\times AE$ will be pr; and we ſhall have the following equation, to wit, $qx+ps-sx=pr$; whence $qx-sx=pr-ps$, and $x=\dfrac{pr-ps}{q-s}$; whence we have the following proportion; as $q-s$ is to $r-s$ ſo is p to x: but $q-s$ is $AG-AC$ or CG, and $r-s$ is $AE-AC$ or CE, and p is AB, and x is AN; therefore as CG is to CE ſo is AB to AN: thus is the point N, and conſequently the poſition of the line $KLMN$ determined. But for a more elegant conſtruction of this problem; ſince CG is to CE as AB to AN, it follows *dividendo*, that $CG-CE$ or EG is to CE as $AB-AN$ or BN is to AN, and inverſely, that CE is to EG as AN to NB; therefore the line AB is divided in N or EF in L in the ſame proportion as CG is divided in E: whence it follows, that if the line CH be drawn, it will cut the line EF in L; for the ſimilar triangles ELC and FLH will give EL to LF as CE to FH, that is, as CE to EG. *Q. E. I.*

Otherwiſe thus: draw the line CH cutting EF in O, and the triangles CGH, CEO will be ſimilar, and CG will be to CE as GH to EO; but CG is to CE as AB to AN, that is, as GH to EL; therefore EL is equal to EO; therefore the line KN drawn through the point L or O, parallel to AG will determine the point N in the line AB; and AK and BM will be the rectangles ſought.

Now if we caſt about for a ſynthetical demonſtration of this conſtruction, we ſhall eaſily find it. For in the rectangle $CGHD$ the two complements EK and DL are equal, by the 43d propoſition of the firſt book of the Elements; and if to theſe equal complements we add the two rectangles AL and BM, we ſhall have the three rectangles AL, EK and BM equal to the three rectangles AL, BM and DL: but the two rectangles AL and EK make AK on one ſide, and the two rectangles BM and DL make BL on the other ſide; therefore the ſum of the two rectangles AK and BM is equal to the ſum of the two rectangles AL and BL, that is, to the rectangle AF. *Q. E. D.*

N. B. By the ſolution of this problem it is eaſy to ſee, that we may often trace out analytically the conſtruction of a problem, and then by this conſtruction be directed to a ſynthetical demonſtration of the ſame, which without the conſtruction firſt obtained, would ſcarce have been thought of.

PRO-

PROBLEM 9.

317. *In a triangle whose base and perpendicular are given, to inscribe a right-angled parallelogram whose base shall be part of, or coincident with the base of the triangle, and whose two contiguous sides shall be to each other in a given ratio.*

As in the triangle ABC (*Fig.* 11,) whose base BC and perpendicular AD are given, let it be required to find a point as I in the line AD through which a line as EF being drawn parallel to the base of the triangle, and terminated by it's legs, and the right-angled parallelogram $EFGH$ being compleated, the side EF may be to the side FG in a given ratio. But here it must be observed, that as every triangle has at least two acute angles, that the parallelogram $EFGH$ may fall wholly within the triangle, the angles B and C must both be acute, or rather, neither of them must be obtuse.

SOLUTION.

Call AD p, BC q, and let the given ratio of EF to FG be that of r to s; where p, q, r and s are supposed to be given lines, whether commensurable or incommensurable it matters not: call also AI x, and consequently ID $p-x$; and by similar triangles we shall have the following proportions, to wit, AD to AI as BD to EI, and also as DC to IF, and consequently as BC to EF: since then AD or p is to AI or x as BC or q to EF, we shall have $EF=\frac{qx}{p}$, or a fourth proportional line to the three lines p, x and q: but EF is to be to FG as r to s; therefore we have the following proportion, to wit, $\frac{qx}{p}$ is to $p-x$ as r to s. Now what we have often observed before in numbers is equally true in lines, to wit, that if four lines be proportionable, a right-angled parallelogram, whose two contiguous sides are the two extremes, will be equal to another right-angled parallelogram, whose two contiguous sides are the two middle terms; or as we usually express it in short, the rectangle of the extremes is equal to the rectangle of the means or middle terms: this is the 16th proposition of the sixth book of the Elements. And it may not be amiss to observe, that by the rectangle of two lines in Geometry must always be understood a right-angled parallelogram whose base is one of the lines, and height the other; and when we speak of the line A multiplied into the line B, we mean no more than the area of such a rectangle whose contiguous sides are A and B. Since then

then $\frac{qx}{p}$ is to $p-x$ as r to s, we shall have $\frac{qsx}{p}=pr-rx$, and $qsx=ppr-prx$, and $prx+qsx=ppr$, and $x=\frac{ppr}{pr+qs}$. Now had the quantities p, q, r, s been given in numbers, the quantity x might have been determined without any further process by computing the quantity $\frac{ppr}{pr+qs}$; but as the quantities p, q, r, s are lines, and perhaps lines incommensurable, this problem is purely geometrical, and must be solved like all others that are so, to wit, by a meer drawing of lines without any computation, which is not allowed in matters purely geometrical. As the case now stands, $pr+qs$ is to pp as r is to x, that is, x is a fourth proportional to two planes and a line; for the rectangles pr and qs are what we call planes, or plain surfaces; whence their sum $pr+qs$ will be a plane, as is also the square pp: therefore to reduce the proportion of these planes, so that x may be found at last a fourth proportional to three lines, one of the rectangles pr or qs, suppose qs, must be so transformed as to have one of it's sides the same with one of the sides of the other rectangle pr, which may be done thus: make r to s as q is to t, that is, find the line t a fourth proportional to the three given lines r, s and q, which may be done by a meer drawing of lines, without any computation, by the 12th proposition of the sixth book of the Elements; and then, since r is to s as q is to t, we shall have $rt=qs$: substitute now rt instead of qs in the expression of x, that is, in the expression $\frac{ppr}{pr+qs}$, and you will have $x=\frac{ppr}{pr+rt}$, or dropping r, you will have $x=\frac{pp}{p+t}$; that is, you will have $p+t$ to p as p is to x; therefore x is a third proportional to the two lines $p+t$ and p, and may be found by the 11th proposition of the sixth book of *Euclid*. This last step furnishes the following construction.

Produce AD *from* D *to* K *below the triangle, so that* BC *may be to* DK *in the given ratio of* r *to* s: *then if you make as* AK *to* AD *so* AD *to* AI, *that is, if you take* AK, AD *and* AI *continual proportionals, you will have* I *the point through which one side* EF *of the parallelogram* EFGH *is to pass: and this side* EF *being once determined, the other sides* FG *and* EH *will easily be determined by drawing perpendiculars to* BC. Q. E. I.

Otherwise without Algebra thus: having produced the line *AD* to *K*, so that *BC* may be to *DK* as *r* to *s*, as above; through *K* draw *LM* parallel to *BC*, and meeting the sides *AB* and *AC* produced in *L* and *M* respectively; let fall *CN* and *BO* perpendiculars to *LM*, and you will

will have a right-angled parallelogram $BCNO$ ſimilar to the parallelogram ſought, ſince BC is to CN as r to s: but the triangles ALM and ABC, wherein theſe ſimilar parallelograms are inſcribed, are alſo ſimilar; therefore the parallelograms $EFGH$ and $BCNO$ are ſimilar parts of ſimilar triangles; therefore all ſimilar lines in theſe figures will be proportionable; therefore AK the altitude of the triangle ALM, will be to AD it's exceſs above the altitude of the parallelogram inſcribed in that triangle, as AD the altitude of the triangle ABC, is to AI it's exceſs above the altitude of the parallelogram ſimilarly inſcribed in that triangle; therefore AK, AD, AI are continual proportionals. *Q. E. I.*

A ſynthetical demonſtration of the foregoing conſtruction may be that which follows. We are to demonſtrate that if the lines AK, AD, AI be taken in continual proportion, the parallelogram $EFGH$ will be ſuch that EF will be to FG as r to s; which may be done thus:

Since by *hypotheſis* AK is to AD as AD is to AI, it follows by converſion of proportion that AK is to KD as AD is to DI, and by permutation that AK is to AD as KD to DI; but AK is to AD as AD to AI by the ſuppoſition, or as BC to EF; therefore BC is to EF as KD to DI; and again by permutation BC is to KD as EF to ID; but BC is to KD as r to s by the conſtruction; therefore EF is to ID or FG as r to s. *Q. E. D.*

N. B. If the parallelogram $EFGH$ is intended to be a ſquare, EF muſt then be equal to FG, and conſequently r to s, and BC to DK.

SCHOLIUM.

For the better underſtanding of the foregoing, and many other geometrical problems, it muſt be obſerved, that *In Geometry three diſtinct ſorts of magnitudes are conſidered, to wit, lines, which have length only, and are ſaid to have but one dimenſion; ſurfaces, which have length and breadth only, and therefore are ſaid to have two dimenſions; and ſolids, which have length, breadth and thickneſs, and ſo are ſaid to have three dimenſions. Thus the ends of lines are points, the edges of ſurfaces are lines, and the outſides of ſolids are ſurfaces.*

Should we abate ever ſo little of the rigour of theſe definitions, theſe different ſorts of magnitudes, how diſtinct ſoever they may otherwiſe ſeem to be, would immediately be confounded one with another: ſhould we allow ever ſo little length to our points, or breadth to our lines, or thickneſs to our ſurfaces, wherein would a point differ from a ſhort line, or a line from a narrow ſurface, or a ſurface from a thin ſolid? But by being conſidered as above, they are not only kept diſtinct one from another, but even rendered heterogeneous one to another, ſo as to be incapable

pable of any comparison one with another by way of proportion. A line may be compared with a line, or a surface with a surface, or a solid with a solid; but a line must by no means be compared with a surface or with a solid, any more than a surface with a solid; because proportion is an affection of homogeneous quantities only, as is evident from the 3d and 4th definitions of the fifth book of *Euclid:* so that to ask what proportion a foot in length bears to a square foot, or a square foot to a cubic foot would be altogether as absurd as to ask what proportion a foot in length bears to an hour in time, or an hour in time to a pound in weight. Therefore whenever four quantities are proportionable, they must either be all homogeneous quantities, or at least the two last terms, though heterogeneous to the two former, must however be homogeneous one to the other: if the third term be a line, the fourth must be a line; if the third be a surface, the fourth must be a surface; if the third be a solid, the fourth must be a solid, and so on.

It must be further observed, that if these letters A, B, C, D, E, F, G represent so many distinct lines, any two of them, as AB, standing together like two factors representing a product in Arithmetic, are in Geometry used to represent the area of a right-angled parallelogram whose two sides are A and B, as was hinted above, and so constitute what is called a plane: any three of them, as ABC, represent the solid content of a parallelepiped whose three dimensions are A, B and C, and so constitute what is called a solid. But if any one of these letters as A, expresses not a line, but a number, then ABC represents not a quantity of three, but of two dimensions; for it signifies the area BC multiplied by the number A, or taken as often as the number A expresses. But these geometrical quantities are sometimes also represented fractionwise; and in such cases the quantity represented must be esteemed a line, plane, or solid, according as the numerator contains one, two, or three dimensions more than the denominator. Thus $\frac{AB}{C}$ signifies a line; for properly speaking, in Geometry $\frac{AB}{C}$ signifies a fourth proportional to the three lines C, B and A: since then the line C is to the line B as the line A is to $\frac{AB}{C}$, and since the third term A is a line, the fourth $\frac{AB}{C}$ must be so too, otherwise the fourth term would not be homogeneous to the third, as (from what has been said) it ought to be. Again, the quantity $\frac{ABC}{DE}$ signifies a line; for as the plane DE is to the plane BC,

so

ſo is the line A to the line $\frac{ABC}{DE}$. Laſtly, $\frac{ABCD}{EFG}$ ſignifies a line; for as the ſolid EFG is to the ſolid BCD, ſo is the line A to the line $\frac{ABCD}{EFG}$. If the numerator contains two dimenſions more than the denominator, the quantity repreſented is a plane: thus $\frac{ABC}{D}$ is a plane; for as the line D is to the line C, ſo is the plane AB to the plane $\frac{ABC}{D}$: thus again $\frac{ABCD}{EF}$ ſignifies a plane; for as the plane EF is to the plane CD, ſo is the plane AB to the plane $\frac{ABCD}{EF}$. If the numerator contains three dimenſions more than the denominator, the quantity repreſented is a ſolid: thus $\frac{ABCD}{E}$ ſignifies a ſolid; for as the line E is to the line D, ſo is the ſolid ABC to the ſolid $\frac{ABCD}{E}$: thus again, $\frac{ABCDE}{FG}$ ſignifies a ſolid; for as the plane FG is to the plane DE, ſo is the ſolid ABC to the ſolid $\frac{ABCDE}{FG}$. Laſtly, if SP, QT and QR be lines, as in the ſcheme annexed, (*Fig.* 12,) the quantity $\frac{SP^2 \times QT^2}{QR}$ will be a ſolid; for as the line QR is to the line QT, ſo will the ſolid $SP^2 \times QT$ be to the ſolid $\frac{SP^2 \times QT^2}{QR}$. See *Newton*'s *Principia, prop.* 6. *cor.* 1. *lib.* 1.

If the numerator and denominator contain an equal number of dimenſions, the quantity thus repreſented will be a number, and no geometrical quantity: thus $\frac{A}{B}$ ſignifies a number, as doth $\frac{AB}{CD}$ and $\frac{ABC}{DEF}$; for (to inſtance in this laſt caſe) as the ſolid DEF is to the ſolid ABC ſo is the number 1 to the number $\frac{ABC}{DEF}$.

If the number of dimenſions in the numerator exceeds the number of dimenſions in the denominator by more than three, or if the denominator hath in it more dimenſions than the numerator, Geometry, properly ſpeaking, has no names for ſuch quantities. But yet if the numerator contains four dimenſions more than the denominator, ſuch a quan-

tity muſt always be looked upon as homogeneous to any other of four dimenſions; and ſo of the reſt.

Theſe conſiderations will be often found of great uſe in detecting errors in a calculation: for if a quantity which in concluſion ought to come out a line, be found to be a plane, a ſolid, or a quantity of any other dimenſions; or if in any part of the operation a quantity be found repreſented by a line, and another of the ſame kind repreſented by a plane, a ſolid, a number, &c, it is an infallible argument that either ſome miſrepreſentation was made at the beginning, or ſome miſtake in the proceſs of the calculation. But if a line be repreſented by unity, (as is ſometimes done to render a calculation leſs perplexed,) all this order and uniformity vaniſhes at once; and it will not be uncommon to ſee one and the ſame ſpecies repreſenting ſometimes a number, ſometimes a line, ſometimes a plane, and ſometimes a ſolid, as the compariſon requires: thus if A ſignifies properly a line, it may be made to ſignify a number by imagining the line repreſented by unity ſtanding under it; or it may be uſed to ſignify a plane whoſe altitude is A, and whoſe baſe is the line repreſented by unity; or laſtly, it may be uſed to ſignify a ſolid whoſe altitude is A, and whoſe baſe is the ſquare of the line repreſented by unity. Neither will it be uncommon in ſuch caſes to find quantities ſeemingly heterogeneous equated to each other: as if xx ſhould be found equal to $5x$, it muſt not in Geometry be underſtood as if the area of the ſquare whoſe ſide is x was equal to 5 times the length of the line x, but that the ſquare of x is equal to 5 times the area of a parallelogram whoſe altitude is x, and whoſe baſe is the line repreſented by unity; or (which is the ſame thing) that the ſquare of x is equal to the area of the parallelogram whoſe altitude is x, and whoſe baſe is the line 5; for if an unit repreſents any line, the number 5 will repreſent another equal to 5 times the former.

This way of repreſenting quantities of one ſort by thoſe of another is not at all amiſs, provided that the repreſentative quantities be always taken in the ſame proportion to each other as thoſe they repreſent, I mean in the ſame computation: ſo that in the ſolution of any problem we are abſolutely at liberty to repreſent any one quantity of any one ſort whatever by any other quantity of any other ſort whatever, let it's real magnitude be what it will, provided that all other quantities of the ſame kind be proportionably repreſented. Thus in Mechanics and mechanical Philoſophy, what is more common than to find times repreſented by proportionable lines, and even rectilinear ſpaces by proportionable planes? The reaſon whereof is, that numbers and geometrical magnitudes are very often more eaſy to be compared than are the quantities repreſented by them: and thus we find the proportions of time, ſpaces, forces, motions,

tions, velocities, &c, which otherwise would not be so easily discovered.

PROBLEM 10. (*Fig.* 13.)

318. *Let* AB *and* CD *be two turrets, and* BD *a horizontal line passing from the foot of one to the foot of the other: It is required, having given in numbers the heights and horizontal distance of these two turrets, to find a place as* E *in the line* BD, *upon which the foot of a ladder being fixed, the ladder may equally serve to reach the top of each turret; or (which amounts to the same thing) let it be required to find a point as* E *in the line* BD, *from whence the lines* EA *and* EC *being drawn, shall be equal one to the other.*

The geometrical effection of this problem is very easy: for if AC be bisected in F, and the line FE be drawn perpendicular to AC, that line shall meet the line BD in the point E required. For the triangles EFA and EFC will have two sides and the angle between them in one, equal to two sides and the angle between them in the other; the side EF will be common to both triangles, the sides FA and FC are equal by construction, and the angles EFA and EFC are both right ones; and therefore the hypotenuses EA and EC will be equal. But as in this problem the heights AB and CD are given in numbers, as also the horizontal distance BD, it is expected that the two segments BE and ED be determined in numbers.

SOLUTION.

Call AB p, CD q, BD r, BE x, and ED $r-x$; and you will have, 1st $AB^2+BE^2=AE^2$ by the 47th of the first book of the Elements; that is, $pp+xx=AE^2$. 2dly you will have $ED^2+DC^2=EC^2$, that is, $rr-2rx+xx+qq=EC^2$; but EA^2 and EC^2 are equal *ex hypothesi*; therefore you will have $pp+xx=rr-2rx+xx+qq$; whence x or $BE=\frac{rr+qq-pp}{2r}$; subtract this value of BE from r, the value of BD, and you will have $ED=\frac{rr-qq+pp}{2r}$, both which expressions are included in the following canon.

To the square of the given horizontal distance add the square of the height of one turret, and subtract the square of the height of the other; divide the remainder by twice the horizontal distance, and the quotient will be the distance of the point sought from the foot of that tower the square of whose height was subtracted. Q. E. I.

As for example; let AB or p, the height of the higher tower be 39 yards, and let CD or q, the height of the lower, be 25 yards, and let

BD or r, the horizontal distance, be 112 yards: then will BE or $\frac{rr+qq-pp}{2r}$ be 52 yards; whence the other segment ED will be 60 yards, and the conditions of the problem will be answered; for we shall have AB^2+BE^2 or $AE^2=1521+2704$ or 4225, and CD^2+DE^2 or $CE^2=625+3600$ or 4225: therefore $AE^2=CE^2$, and $AE=CE=65$.

SCHOLIUM.

If in this scheme the line CD vanishes, that is, if C coincides with D, we shall have $q=0$, and the problem will be reduced to this, *viz. Having given one leg of a right-angled triangle, together with the sum of the other leg and the hypotenuse, to find that other leg and the hypotenuse separately*. For in this case we shall only have given AB and BD, whereof AB is one leg of the right-angled triangle ABE, and BD is the sum of BE and ED or BE and EA, that is, the sum of the other leg and the hypotenuse; whereof BE will be $\frac{rr-pp}{2r}$, and AE will be $\frac{rr+pp}{2r}$.

PROBLEM II. (*Fig.* 14.)

319. *It is required, having given in numbers the* radius *of any circle, together with the tangent of any arc thereof that is less than half a quadrant*, (the necessity of which limitation will be shewn hereafter,) *to find the tangent of twice that arc.*

Let A be the center of any circle BCD, whereof let BC be any arc less than half a quadrant, and let BCD be the double of that arc; let BEF be a tangent to the circle in the point B, and join AB, ACE, and ADF; and BE in Trigonometry is called the tangent, and AE the secant of the arc BC, as is BF the tangent and AF the secant of the arc BD. Let it be required then, having given in numbers the *radius* AB and the tangent BE of the arc BC, to find BF the tangent of twice that arc.

N. B. If the line BF was only required, nothing would be more easy than a geometrical construction of this problem: it is only taking the angle EAF equal to the angle EAB by the 23d of the first book of the Elements, and the line BF will be determined as to it's geometrical quantity. But what is here required is to determine the line BF in parts of the *radius*, and consequently in numbers, as all other tangents in trigonometrical tables are expressed.

SOLU-

SOLUTION.

Call AB, the given *radius* r, BE, the given tangent t, and BF the unknown tangent x; then will EF be $x-t$, and AF, the hypotenuse of the right-angled triangle ABF, will be $\sqrt{r^2+x^2}$. Now since the line AE bisects the angle BAF; the two segments of the base, to wit BE and EF, will be in the same proportion to each other as the adjacent legs AB and AF, by the 3d of the sixth book of the Elements; that is, BE will be to EF as AB to AF; whence, by art. 16 paragraph 6, BE^2 will be to EF^2 as AB^2 to AF^2, that is, according to our notation, tt will be to $xx-2tx+tt$ as rr to $rr+xx$; multiply the means and extremes of these four proportionable numbers, and you will have $r^2x^2-2r^2tx+r^2t^2=r^2t^2+t^2x^2$; throw away r^2t^2 from both sides, and transpose t^2x^2, and you will have $r^2x^2-t^2x^2-2r^2tx=0$; whence x or $BF=\frac{2r^2t}{r^2-t^2}$; therefore AB is to BF as r is to $\frac{2r^2t}{r^2-t^2}$, or as 1 is to $\frac{2rt}{r^2-t^2}$, or as r^2-t^2 is to $2rt$; which is as much as to say, that *If in any circle whose* radius *is* r, t *be the tangent of any arc, the tangent of twice that arc may be found by saying, as* rr — tt *is to* 2rt, *so is* r *the* radius *of the circle, to* $\frac{2r^2t}{r^2-t^2}$ *the tangent sought.* If the proportion of AB to AF, that is, of the *radius* to the secant of twice the arc be required, that will be the same with the proportion of BE to EF, as above: but $BE=t$, and $EF=x-t=\frac{2r^2t}{r^2-t^2}-t=\frac{r^2t+t^3}{r^2-t^2}$; therefore AB is to AF, or BE to EF as t is to $\frac{r^2t+t^3}{r^2-t^2}$, or as 1 to $\frac{r^2+t^2}{r^2-t^2}$, or as r^2-t^2 is to r^2+t^2; but $r^2+t^2=AB^2+BE^2=AE^2$ equal to the square of the secant of the arc BC: call this secant s, and then AB will be to AF as r^2-t^2 is to s^2; but it was found before that AB was to BF as r^2-t^2 was to $2rt$; therefore *If* t *and* s *be the tangent and secant of any arc of a circle whose* radius *is* r, *the* radius, *tangent and secant of twice the arc will be as* r^2-t^2, $2rt$ *and* ss, *or as* rr—tt, 2rt *and* rr+tt *respectively.* Q. E. I.

The Moderns have enlarged this theorem very much, so as to express the proportion of the *radius*, tangent and secant of 3 times, 4 times, 5 times the original arc, &c. But it will be impossible for the learner to

see

fee the beauty of that theorem till he underſtands the method of reſolving the powers of a binomial; of which more in another place.

SCHOLIUM.

The reaſon of the limitation preſcribed in the foregoing problem, to wit, that the arc *BC* muſt be leſs than half a quadrant, is obvious; for ſhould the arc *BC* be ſuppoſed half a quadrant, it's double *BD* would be a quadrant, and the angle *BAD* would be a right angle: but the angle *ABE*, which the tangent makes with a *radius* paſſing through the point of contact, is a right angle by the ſixteenth of the third book of the Elements; therefore in this caſe, the two internal angles *ABE* and *BAD* are equal to two right ones; whence the lines *AD* and *BE* are parallel to each other, and conſequently can never meet, how far ſoever they may be produced; therefore the tangent of a quadrantal arc is infinite. If the arc *BD* be taken leſs than a quadrant, it is certain that the lines *AD* and *BE* will, if produced far enough, meet in ſome point as *F*, and that it is this point of concourſe *F* that determines the line *BF* to be the tangent of the arc *BD*. It is certain too that the nearer the arc *BD* approaches to a quadrant, the further muſt the lines *AD* and *BE* be produced before they can meet, and the greater will be the tangent *BF*; therefore when *BD* becomes a quadrant, and conſequently *AD* parallel to *BE*, the point *F* may now be ſuppoſed to have gone off *ad infinitum*, and the tangent *BF*, as well as the ſecant *AF* to become infinite. This is the reaſon why in trigonometrical tables the tangent and ſecant of a quadrantal arc of ninety degrees are not expreſſed like the reſt in numbers, but by the word *infinite*.

It is not improbable but that my reader may be ſomewhat ſtartled at this word infinite: nor can it be denied but that there are ſome difficulties attending the ſuppoſition of infinite quantities, which if not removed, are apt very much to perplex and confound narrow minds; but of theſe I ſhall ſpeak more at large upon a more proper occaſion: at preſent I ſhall only enter ſo far into the notion of infinity as it relates to a certain doctrine of another kind I am here going to advance, and which, I fear, my reader will find harder of digeſtion than any thing he has hitherto met with, unleſs he will ſuffer himſelf to be perſwaded, as well upon this, as upon almoſt all other occaſions, when he is reading the words and ſenſe of another, to apply himſelf wholly to them, to view every thing in the light it is placed in, and not to form a judgement of things from his own narrow conceits and little prejudices on one hand, or from high flown, metaphyſical, and perhaps chimerical notions on the other, which ſerve but to bewilder his underſtanding, and to draw off his thoughts from the main ſubject, which ought to take up his whole atten-

attention. Geometrical truths are plain ſimple truths, and whoever would ſee them in the cleareſt light, ought to view them in the ſimpleſt: for it is here as in Optics, where all light let in upon an object beſides what is proper to give a diſtinct view of it, is not only ſuperfluous, but tends rather to obſcure than to illuſtrate the object. But to come to the point.

The poſition I am here going to advance is, that *There are two paſſages from affirmation to negation, to wit, through nothing and through infinity:* the former is obvious to almoſt every one, whilſt the latter is open but to few, except ſuch as are converſant in the ſublimer parts of Geometry and Mathematics: but I hope, taking occaſion from the foregoing problem, to make good even this part of my aſſertion to the meaneſt proficient, provided he will not be wanting to himſelf, but aſſiſt with his attention. (*See Fig.* 15.)

Let then A be the center of the circle $BGHK$, divided into four equal quadrants BG, GH, HK, KB by the two croſſing diameters BH and GK, and let BL be a tangent infinitely produced both ways from B, the points L and G being both ſuppoſed on the ſame ſide of the diameter BH: let D be a moveable point in the circumference; and as it moves, let it be ſuppoſed always to carry the infinite or indefinite line ADF along with it; which line is ſuppoſed, during it's motion, to turn upon the center A, and to meet the tangent in F. Let now the point D be ſuppoſed firſt of all to be in the quadrant BG, moving from B towards G; then it is manifeſt that the nearer D approaches to G, the greater will be the diſtance BF between the fixed point of contact B and the moveable interſection F. When D coincides with G, the diſtance BF will be infinite, as was ſhewn before in this article, the line AD being parallel to the tangent BL. Let us now ſuppoſe the point D to paſs through G into the quadrant GH; then it is plain that the two lines AD and BL will be ſo far from meeting on that ſide, that they will actually diverge; but it appears however from this divergency, that if the line AD be indefinitely produced the other way from A, it will then meet the tangent again in ſome point as F, and that the point of concourſe F will now have changed ſides with reſpect to the fixed point B, ſo that if BF was conſidered as affirmative whilſt the point F fell on the ſame ſide of B with the point L, the diſtance BF ought now to be looked upon as negative when F falls on the contrary ſide: therefore whilſt the point D moved out of the quadrant BG through the point G into the quadrant GH, the diſtance BF paſſed from an affirmative through infinity into a negative; and the nearer the point D approaches the point H, the leſs will be the negative diſtance BF. When D coincides with H, the negative diſtance BF will vaniſh, or become equal to nothing; and when D has

has paſſed through H into the quadrant HK, the point F will now again change ſides, and fall on the ſame ſide of B with the point L: therefore the diſtance BF will now again become affirmative, to wit, by paſſing from a negative through nothing into an affirmative. Let the point D continue it's motion from H to K; and when D arrives at K, the diſtance BF will again become infinite; and when D has paſſed through K, the point F will again fall on the negative ſide of the point B, and the diſtance BF will be negative, having paſſed into that ſtate from an affirmative through infinity. Thus will BF continue negative till D has paſſed through B again into the firſt quadrant BG; for in that time BF will have paſſed from a negative through nothing into it's firſt ſtate of affirmation.

Here then we have a pregnant inſtance of both tranſitions in the tangent BF, having changed it's ſign no leſs than four times during one revolution of the point D, to wit, twice by paſſing through nothing, and twice by paſſing through infinity.

Let us enquire in the next place what changes the ſecant AF paſſes through in the ſame time. In the firſt place then we are to take notice that the ſecant AF never vaniſhes or paſſes into a ſtate of nothingneſs, as doth the tangent BF: for the ſecant AF, when it is leaſt, is ſtill equal to the *radius* AB, as in the caſes when D exiſts in B and in H; in other caſes it is greater; and when D paſſes through G or K, the ſecant AF paſſes through infinity, and changes it's ſtate from an affirmative to a negative, and *vice verſa*, becauſe then the point F changes ſides with reſpect to the fixed point A in the line AF. If therefore the ſecant AF be conſidered as affirmative whilſt the point D paſſes through the ſemicircle KBG, it muſt be looked upon as negative whilſt the ſame point D paſſes through the oppoſite ſemicircle GHK: therefore the ſecant AF changes it's ſign twice during one revolution of the point D, and both times by paſſing through infinity.

From what has been demonſtrated concerning the tangent BF and the ſecant AF we may obſerve

1ſt, That when a quantity paſſes from a ſtate of affirmation through nothing into a ſtate of negation, it paſſes from an infinitely ſmall affirmative to an infinitely ſmall negative, and it's negation increaſes afterwards: but when a quantity paſſes from a ſtate of affirmation through infinity into a ſtate of negation, it paſſes then from an infinitely great affirmative to an infinitely great negative, and it's negation diminiſhes afterwards.

2dly, We may obſerve that a quantity infinitely great is as ambiguous with reſpect to it's ſtate of affirmation or negation, as is a quantity evaneſcent or nothing. When the point D coincided with the point G, the lines AD and BL were parallel, and conſequently could have no more

more inclination to meet on one ſide than on the other; therefore though it was very proper to ſay that in this caſe BF was infinite, yet no one could undertake to ſay whether it was affirmative or negative; no reaſon could be given for one ſide of the queſtion that might not as well be alledged for the other. Therefore the tranſition from an infinitely great affirmative to an infinitely great negative ceaſes now to be a myſtery, ſince it is no tranſition; for at the ſame inſtant of time that ſuch a quantity is an infinite affirmative, it is alſo an infinite negative, without any ſucceſſion of time or motion.

3dly, It may be obſerved, that if in the caſe of an arc leſs than a quadrant, the tangent and ſecant be both affirmative, (and ſo they are always conſidered,) in an arc greater than a quadrant and leſs than two quadrants, they will both be negative; in an arc greater than two quadrants and leſs than three, the tangent will be affirmative and the ſecant negative; and laſtly in an arc greater than three quadrants and leſs than four, the ſecant will be affirmative and the tangent negative: thus by a compariſon of the ſigns of the tangent and ſecant of any arc whatever, may the affection of that arc be determined. This, if duly conſidered and underſtood, will take off the limitation the foregoing problem lay under: for ſuppoſe now the arc BC in the ſcheme thereunto belonging *(Fig. 14,)* ſhould be taken equal to, or greater than half a quadrant, all the conſequence will be that the tangent of twice that arc will be found infinite or negative; becauſe twice the arc will be equal to, or greater than a quadrant.

4thly, We may obſerve, that as the demonſtration of what is here delivered depended chiefly upon making the lines AF and BF paſs through all poſſible degrees of magnitude, ſo many other properties and relations of flowing quantities may be diſcovered this way, which would ſcarce have been perceived, had thoſe quantities been ſuppoſed to continue fixed and determinate: and upon this variation of quantities from one degree of magnitude to another, is founded one of the fineſt inventions in all the modern *Analyſis*, I mean the doctrine of Fluxions. But it is high time to proceed to another ſhort inſtance or two for a further confirmation of the doctrine here advanced.

For that purpoſe, let BCD *(Fig. 16)* be an arc of a circle whoſe middle point is C, and let A be the center of curvature; that is, if the arc BCD was compleated into an entire circle, let A be the center of that circle; and through the points A and C let the line AC be drawn, and indefinitely produced both ways. This done, without changing either the place of the middle point C, or the length of the arc BCD, let that arc be ſuppoſed to unbend itſelf a little, ſo as to become a part of a larger circle if compleated, than before: then it is plain that the cen-

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ter A will recede further from the point C, and the length of the line CA will be greater than before, as being the ſuppoſed *radius* of a greater circle. Let us ſuppoſe the arc BCD ſtill to unbend itſelf more and more by degrees till it becomes a ſtreight line, and the diſtance CA growing by this means greater and greater, will at laſt become infinite; for a ſtreight line cannot be conſidered as an arc of any circle but an infinite one: the greater the *radius* of any circle is, the leſs will be the curvature of any particular arc of that circle, and *vice verſa*; and therefore when the length of the *radius* becomes infinite, that is, greater than any aſſignable length whatever, the curvature of the arc will be infinitely ſmall, that is, leſs than any aſſignable curvature whatever; and therefore ſuch an arc in ſuch a caſe will differ nothing from a ſtreight line of the ſame length. Let us now ſuppoſe the arc BCD, having thus unbent itſelf by degrees into a ſtreight line, to bend itſelf the other way, which is but a continuation of our former ſuppoſition, and the center of curvature A muſt now neceſſarily be found on the other ſide the point C; and therefore if the diſtance CA in the former caſes was affirmative, it will now be negative, the point A having changed ſides with reſpect to the fixed point C. It is further evident alſo, that this ſtate of negation was attained by a paſſage through infinity; and that this diſtance or *radius* when infinite, had an equal right to be ſtiled affirmative or negative: and this is all I wanted to illuſtrate by this inſtance.

Before I proceed to my next inſtance, which ſhall be the laſt I ſhall trouble my reader with of this kind, it will be proper to premiſe the following lemma, to wit,

That *if* $\frac{1}{v}$ *be any fraction whoſe numerator is* 1 *or any other finite number whatever, and whoſe denominator* v *is an infinitely ſmall fraction expreſſed after the manner of a whole number; I ſay then that ſuch a fraction* $\frac{1}{v}$ *will expreſs a number infinitely great.* For whilſt the numerator 1 continues the ſame, the leſs the denominator v is, the greater will be the fraction: if v be 1, or $\frac{1}{10}$, or $\frac{1}{100}$, or $\frac{1}{1000}$, &c, $\frac{1}{v}$ will be 1, or 10, or 100, or 1000, &c; therefore if the denominator v be infinitely ſmall, the fraction $\frac{1}{v}$ will be infinitely great. *Q. E. D.*

Or thus: Every fraction is equal to the quotient ariſing from dividing the numerator by the denominator; and every quotient ſhews how often the diviſor is contained in the dividend; therefore every improper fraction ſhews how often the denominator is contained in the numerator.

Thus

Thus the improper fraction $\frac{12}{3}$ or 4 ſhews that the denominator 3 is 4 times contained in the numerator 12: whence it follows that if the denominator of a fraction be infinitely ſmall whilſt the numerator is finite, the fraction muſt be infinitely great, becauſe an infinitely ſmall quantity is contained in a finite one an infinite number of times. *Q. E. D.*

Or thus: As v is to 1 ſo is 1 to $\frac{1}{v}$; therefore *e converſo* $\frac{1}{v}$ will be to 1 as 1 is to v; but 1 is infinitely greater than v, becauſe v is infinitely ſmall; therefore $\frac{1}{v}$ will be infinitely greater than 1, or (which is the ſame thing) the fraction $\frac{1}{v}$ will repreſent a number infinitely great. *Q. E. D.*

This being allowed, let $\frac{1}{v}$ repreſent any fraction whoſe numerator 1 is always the ſame, but whoſe denominator v is a variable quantity: then it is plain that ſo long as the denominator v is finite and affirmative, the fraction $\frac{1}{v}$ will be ſo too. Let us now ſuppoſe the denominator v to diminiſh by degrees till it becomes equal to nothing; and from what has been ſaid, the fraction $\frac{1}{v}$ will increaſe by degrees till it becomes infinite. Let the denominator v be ſunk lower ſtill, ſo as now to become leſs than nothing or negative, and the fraction $\frac{1}{v}$ will alſo become negative, ſince the quotient of an affirmative quantity divided by a negative one is negative: thus if v be $\frac{-1}{1000}$, or $\frac{-1}{100}$, or $\frac{-1}{10}$, or -1, the fraction $\frac{1}{v}$ will be -1000, or -100, -10, or -1. It is plain alſo that the fraction $\frac{1}{v}$ attained this negative ſtate by having paſſed through infinity: whence it follows that theſe two tranſitions, through nothing and through infinity, are ſo far from being inconſiſtent with, or contradictory to one another, that they are neceſſarily conſequent of, and actually depend upon one another; and that the ambiguous ſtate of an infinite quantity in reſpect to it's affirmation or negation is ſo far from being a myſtery, that it is a neceſſary conſequence of the ambiguous

 ſtate

ftate of nothingnefs in a like refpect. It follows alfo from what has been faid, that to be lefs than nothing, or to be greater than infinite, are but two different appellations of the fame thing, that is, of a negative quantity; or that if there be any difference, it lies only in our manner of conception: for if a negative quantity has gained that ftate from it's having paffed through nothing, we fay it is lefs than nothing; but if it has got it's negation by paffing through infinity, we then fay it is greater or more than infinite.

But there are fome quantities which cannot defcend lower than nothing, or afcend higher than infinity: thus the quantity vv can never be lefs than nothing, though v may; and $\frac{1}{vv}$ can never be greater than infinite, though $\frac{1}{v}$ may.

If this laft inftance be well underftood, it will enable us to look back and to examine fomewhat more narrowly into the canon for folving the laft problem, which is this; that if r be the *radius* of any circle, and t be the tangent of any arc, the tangent of twice that arc will be $\frac{2r^2t}{r^2-t^2}$. Now if this canon be juft, the tangent of a quadrantal arc ought to be infinite, and the tangent of an arc greater than a quadrant and lefs than two quadrants will be negative. Let us fee how this follows from the canon: in order whereto, let BC, in the fcheme of the eleventh problem (*Fig.* 14.) be half a quadrant, and the angle BAE will be half a right one, and confequently the other angle BEA will be fo too; therefore the angles BAE and BEA will be equal, as will their oppofite fides BA and BE; therefore if BC be half a quadrant, it's tangent t will be equal to r the *radius*: fubftitute therefore r inftead of t in the foregoing canon, and you will have the tangent of a quadrantal arc equal to $\frac{2r^3}{rr-rr}=\frac{2r^3}{0}$; but $\frac{2r^3}{0}$ expreffes an infinite quantity by the laft inftance, and therefore the tangent of a quadrantal arc is infinite. If t be the tangent of an arc greater than half a quadrant, t will be greater than r, and $rr-tt$ will be negative; and therefore $\frac{2rrt}{rr-tt}$, the tangent of an arc greater than a quadrant will be negative: and after the fame manner may the other cafes be purfued.

Thefe confiderations point out to us, in fome meafure, the ufe of the foregoing fpeculations. There are few theorems or problems but what admit of divers cafes; and for an author or a mafter to run over all thefe

cafes,

cases, and to furnish rules and demonstrations for them all, would be very troublesome to a learner, and burdensome to his memory; whereas by the help of the foregoing principles, if one extreme case of any proposition be known, all the rest are easily mastered without the formality of geometrical demonstrations, only by observing how in passing from one case to another, the quantities therein concerned pass from affirmation to negation, and *vice versa*. Thus in Catoptrics and Dioptrics, if the properties of a convex surface be once known, those of a plain and concave surface will easily be obtained by increasing the *radius* of curvature to an infinite, or to a more than infinite length. Thus in Conic Sections, if the properties of the *ellipsis* be well known, those of the *parabola* will easily be deduced from them; and even those of the *hyperbola* that have any affinity with the properties of the *ellipsis*, only by retaining one vertex and the nearer *focus*, and supposing the rest to run off *ad infinitum* in the case of the *parabola*, or to a distance more than infinite in the case of the *hyperbola*; provided that in this last case the conjugate *axis* be made possible by changing the sign of it's square. When Mathematicians, I say, of any experience have made themselves masters of these extreme cases, they rarely give themselves any trouble concerning the rest, as well knowing that they have them in their power whenever they shall see occasion to examine into them: but then for such not to acquaint their pupils, as soon as possible, with this secret, is (I think) laying burdens upon the necks of their disciples which they themselves are scarce able to bear. For my own part, I never see young beginners sweating and fretting about arithmetical complements in logarithms, or about the falling of perpendiculars in the resolution of some cases of spherical triangles, matters wherein little more than bare numbers are concerned, but I am in pain for them; and cannot think it would be lost labour for such Gentlemen to acquaint themselves with the nature of affirmative and negative quantities as they stand in opposition to each other, even though they may not find themselves at leisure to enter into the more abstruse parts of Algebra.

PROBLEM 12.

320. *To divide a given line into two such parts, that the rectangle thereof may be equal, if possible, to a given square.*

SOLUTION.

Let a be the line to be divided, let b be the side of the given square, and let x and $a-x$ be the two segments sought; and we shall have this equation, $ax-xx=bb$, or changing all the signs, $xx-ax=-bb$; whence,

whence, filling up the square, we have $xx - ax + \frac{aa}{4} = \frac{aa}{4} - bb$, and $x - \frac{1}{2}a = \pm\sqrt{\frac{aa}{4} - bb}$; therefore $x = \frac{1}{2}a \pm \sqrt{\frac{aa}{4} - bb}$: whence we have the following geometrical construction. (*Fig.* 17.)

Take AB equal to a the line to be divided, and bisecting it in C, draw CD perpendicular to AB by the 11th of the first book of the Elements, and equal to b the side of the given square: then if in the right angle DCA be inscribed the line DE equal to AC, the point E will divide the line AB into the two segments AE and EB required. *Q. E. I.*

For in this construction it is plain that $AC = \frac{1}{2}a$, that $CE = \sqrt{\frac{aa}{4} - bb}$, and consequently that AE is one of the values of x, and EB the other.

N. B. To inscribe a line as DE equal to AC in the angle DCA, is nothing else but to open the compasses to the length of the line AC, and then setting one foot in D as on a center, to make the other cut the line AB in the point E, on the side of the angle DCA.

A synthetical demonstration of the foregoing construction.

By the 5th of the second book of the Elements, AEB, that is $AE \times EB$, together with the square of CE, is equal to the square of AC; but the square of AC is equal to the square of DE by construction, and the square of DE is equal to the square of CD together with the square of CE by the 47th of the first book of the Elements; therefore the rectangle AEB, with the square of CE, is equal to the square of CD with the square of CE; throw away the square of CE from both sides, and you will have the rectangle AEB equal to the square of CD. *Q. E. D.*

From the foregoing construction it appears, that CD the side of the given square must be less than half AB or than AC, since otherwise it would be impossible in the right angle DCA to inscribe the line DE equal to AC.

PROBLEM 13.

321. *To divide a given line into two such parts, that the sum of their squares may be equal to a given square.*

SOLUTION.

Let a be the line to be divided, let b be the side of the given square, and let x and $a - x$ be the two segments sought; then will the sum of their squares be $2xx - 2ax + aa = bb$; whence $2xx - 2ax = bb - aa$, and

and $xx - ax = \frac{bb - aa}{2}$; add the square of half the coefficient to both sides, and you will have $xx - ax + \frac{aa}{4} = \frac{bb - aa}{2} + \frac{aa}{4} = \frac{bb}{2} - \frac{aa}{4}$; whence $x - \frac{a}{2} = \pm\sqrt{\frac{bb}{2} - \frac{aa}{4}}$, and $x = \frac{a}{2} \pm \sqrt{\frac{bb}{2} - \frac{aa}{4}}$, which canon gives the following construction. *(Fig. 18.)*

Let the line to be divided be AB, which bisect in C, and from C set off, towards A or B, CD equal to half the side of the given square; erect CH perpendicular to AB, out of which take CE equal to CD, and $CF = CA$, and draw DE: I say then that if in either of the angles FCA or FCB be inscribed the line FG equal to DE, AG and GB will be the two segments sought. For $FC = \frac{1}{2}a$ *ex hypothesi*, that is, $FC^2 = \frac{aa}{4}$, and FG^2 or $DE^2 = \frac{bb}{2}$; therefore $CG^2 = \frac{bb}{2} - \frac{aa}{4}$, and $CG = \sqrt{\frac{bb}{2} - \frac{aa}{4}}$.

A synthetical demonstration of this construction.

By the ninth of the second book of the Elements, $AG^2 + BG^2 = 2AC^2 + 2CG^2 = 2FC^2 + 2CG^2 = 2FG^2 = 2DE^2 = 4CD^2$ equal to the given square. *Q. E. D.*

From the foregoing construction it appears, that the given square must be greater than half the square of the given line to be divided, and less than the whole square. For first, as the line FG is inscribed within the right angle FCA or FCB, that line FG must be greater than FC, that is, DE must be greater than AC, and DE^2 must be greater than AC^2; but DE^2 equals $2CD^2$, and $AC^2 = \frac{AB^2}{4}$; therefore $2CD^2$ must be greater than $\frac{AB^2}{4}$, and $4CD^2$, or the given square, must be greater than $\frac{AB^2}{2}$. *Q. E. D.*

Again, as the problem requires that AG and BG shall be segments of AB, it is plain that G must lie between A and B; therefore (drawing FA) FG must be less than FA; therefore DE must be less than FA: but CDE and CAF are similar triangles, because CD is equal to CE, and CA to CF; therefore if DE be less than AF, CD must be less than CA, and $2CD$, the side of the given square, must be less than AB; therefore the given square must be less than the square of AB. *Q. E. D.*

PROBLEM 14.

322. *To divide a given line in extreme and mean proportion.*

N. B. *A line is said to be divided according to extreme and mean proportion, when the line and it's two segments are in continual proportion; that is, when the whole line is to it's greater segment as that greater segment is to the less.*

SOLUTION.

Let a be the given line to be divided, whereof let x be the greater segment and $a-x$ the less; then by the foregoing definition a will be to x as x is to $a-x$; and we shall have $xx = aa - ax$, and $xx + ax = aa$, and $xx + ax + \frac{aa}{4} = \frac{5aa}{4}$, and $x + \frac{a}{2} = \pm\sqrt{\frac{5aa}{4}}$, and $x = -\frac{a}{2} \pm \sqrt{\frac{5aa}{4}}$: but $-\frac{a}{2} - \sqrt{\frac{5aa}{4}}$ is a negative root, and all negative segments are excluded by the nature of the problem; therefore $x = +\sqrt{\frac{5aa}{4}} - \frac{a}{2}$, which last step furnishes the following construction. *(Fig. 19.)*

Let AB be the line to be divided, and at right angles to it draw AC equal to half the line AB, and join BC; then set off, from A towards B, AD equal to the excess of BC above AC, and the line AB will be divided according to extreme and mean proportion in the point D.

For since $AB = a$, and $AC = \frac{1}{2}a$, we shall have $BC = \sqrt{\frac{5aa}{4}}$

whence $BC - AC$, or $AD = \sqrt{\frac{5aa}{4}} -$ or x.

Euclid's *synthetical demonstration of the foregoing construction, in the eleventh of the second book of the Elements.*

Produce the line AC both ways, to wit, from C to E, and from A to F, so that AE may be equal to AB, and AF to AD; compleat the squares $AFGD$ and $AEHB$, and let GD produced meet EH in I. This done, since EA is bisected in C, and AF is added to it, we shall have the rectangle $EF \times FA$, together with the square of AC, equal to the square of CF by the sixth of the second book of the Elements: but the rectangle $EF \times FA$ is nothing else but the parallelogram EG; therefore

fore the parallelogram EG with the ſquare of AC will be equal to the ſquare of CF: but as AF equals AD, if to both be added AC, we ſhall have CF equal to $AD+AC$ equal to BC, becauſe $AD=BC-AC$; therefore the parallelogram EG with the ſquare of AC, is equal to the ſquare of BC: but the ſquare of BC is equal to the ſquare of AB and the ſquare of AC put together, by the fortyſeventh of the firſt book of the Elements; that is, to the ſquare AH with the ſquare of AC; therefore the parallelogram EG with the ſquare of AC, is equal to the ſquare AH with the ſquare of AC; neglect on both ſides the ſquare of AC, and you will have the parallelogram EG equal to the ſquare AH; ſubtract from both the common parallelogram DE, and you will have left the ſquare AG equal to the parallelogram IB: therefore by the ſeventeenth of the ſixth book of the Elements, BH or AB will be to AD as AF or AD is to BD. *Q. E. D.*

The two ſegments of a line divided in extreme and mean proportion are incommenſurable to the whole and to one another, as was proved in art. 203, example 2: whence it follows that no number can be divided in extreme and mean proportion, becauſe all numbers whether integral or fractional, are commenſurable.

PROBLEM 15.

323. *To find three lines in continual proportion, having given both their ſum and the ſum of their ſquares.*

SOLUTION.

Let a be the given ſum of the lines, and let b be another line of ſuch a length, that the rectangle ab may be equal to the ſum of the ſquares given: then muſt the line b be found by applying the ſum of the ſquares given to the given line a, according to the fortyfifth of the firſt Element. This application of a given ſurface to a given line is a common phraſe in Geometry, and ſignifies no more than finding the perpendicular altitude of a right-angled parallelogram, which having the given line for it's *baſis*, is equal in area to the ſurface given: it anſwers in Arithmetic to dividing the number repreſenting the ſurface by the number repreſenting the line; and therefore *applicare ad*, when interpreted arithmetically, ſignifies no more than *to divide by*. But to return.

This problem was ſolved in art. 163, where the middle term was found to be $\frac{a-b}{2}$, and the ſum of the extremes $\frac{a+b}{2}$; but for variety's ſake, I ſhall here add another ſolution of the ſame problem as follows.

Since the three lines sought are to be continual proportionals, let their names be x, y and $\frac{yy}{x}$, and we shall have the following equations:

$$\text{Equ. 1st, } xx+xy+yy=ax,$$
$$\text{2d, } x^4+x^2y^2+y^4=abx^2.$$

Divide the second equation by the first, that is, divide $x^4+x^2y^2+y^4$ by x^2+xy+y^2, and the quotient will be x^2-xy+y^2; divide also abx^2 by ax, and the quotient will be bx, and we shall now have

$$\text{Equ. 3d, } xx-xy+yy=bx.$$

Subtract the third equation from the first, and you will have

$$\text{Equ. 4th, } 2xy=ax-bx, \text{ and}$$
$$\text{5th, } y=\frac{a-b}{2}.$$

Hence, of the three lines sought, the mean y is discovered; and since the square of this mean is equal to the rectangle of the extremes, it follows that the rectangle of the extremes will be equal to yy, or to the square of $\frac{a-b}{2}$: but the sum of the extremes is also known; for since the sum of all the three lines is a, if from this sum be subtracted the middle line y or $\frac{a-b}{2}$, there will remain $\frac{a+b}{2}$ for the sum of the extremes. Therefore this problem is now reduced to the twelfth, see art. 320; for it consists now in dividing a given line, as $\frac{a+b}{2}$, into two such parts that the rectangle of those parts may be equal to the square of y, or the square of $\frac{a-b}{2}$: therefore the construction of this problem will be the same, *mutatis mutandis*, with the construction of the twelfth, thus: (*Fig.* 20.)

Draw the line *ABC*, so that *AB* may be equal to $\frac{a}{2}$, and *BC* to $\frac{b}{2}$, and consequently *AC* to $\frac{a+b}{2}$; and you will have *AC* equal to the sum of the extremes. Bisect *AC* in *D*, and draw the perpendicular $DE=y$ or $\frac{a-b}{2}$, or $AB-BC$; and DE^2 will be equal to the rectangle of the extremes. In either of the angles *EDA* or *EDC* inscribe the line *EF* equal to *AD*, and *AF* and *FC* will be the extremes sought. But because the rectangle *ABC* is equal to $\frac{1}{2}a\times\frac{1}{2}b$, or to $\frac{1}{4}ab$, that is, to a fourth

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part of the ſum of the ſquares given, this conſtruction may be better explained from the original *data* of the problem thus:

Draw the line ABC, ſo that AB may be half the ſum of the lines ſought, and that the rectangle ABC may be a fourth part of the ſum of their ſquares: biſect AC in D, and draw DE perpendicular to AC, and equal to $AB-BC$: then if in either of the angles EDA or EDC be inſcribed the line $EF=AD$, the lines AF, DE and FC will be the three lines ſought. *Q. E. I.*

A ſynthetical demonſtration of the foregoing conſtruction.

Here we are to demonſtrate three things; 1ſt, that the lines AF, DE and FC are in continual proportion; 2dly, that their ſum is equal to $2AB$; and laſtly that the ſum of their ſquares is equal to $4ABC$, that is, to four times the rectangle $AB\times BC$.

Firſt then, upon the center D, and with the *radius* DA or DC ſweep the ſemicircle AGC, cutting in G the line FG perpendicular to AC, and join AG, CG, DG; and we ſhall have by the Pythagoric theorem $DE^2+DF^2=EF^2$, and $DF^2+FG^2=DG^2$; but $EF^2=DG^2$, becauſe DG and DA $(=EF)$ are *radii* of the ſame circle; therefore $DE^2+DF^2=DF^2+FG^2$; take away DF^2 from both ſides, and you will have $DE^2=FG^2$, and $DE=FG$. Again, the triangle AGC is a right-angled triangle, as being inſcribed in a ſemicircle; therefore the triangles FGA and FGC are ſimilar triangles by the eighth of the ſixth Element; therefore AF is to FG in one triangle, as FG is to FC in the other; that is, AF, FG and FC are continual proportionals: but DE has been proved equal to FG; therefore the three lines AF, DE and FC are in continual proportion. *Q. E. D.*

2dly, $AF+FC=AB+BC$, and $DE=AB-BC$ by the conſtruction; add equals to equals, and you will have $AF+DE+FC=2AB$. *Q. E. D.*

3dly, from B towards A upon the line BA ſet off BH equal to BC, and then DE, which is equal to $AB-BC$, will be equal to $AB-BH$, that is, to AH. Moreover, ſince $AF+FC=AH+HC$, the ſquare of the former ſum will be equal to the ſquare of the latter; that is, $AF^2+2AFC+FC^2$ will be equal to $AH^2+2AHC+HC^2$: but DE has been proved a mean proportional between AF and FC; and therefore DE^2 will be equal to the rectangle AFC, and $2DE^2$ to $2AFC$: ſubſtitute therefore $2DE^2$ inſtead of $2AFC$, and you will have $AF^2+2DE^2+FC^2=AH^2+2AHC+HC^2$: but DE has already been proved equal to AH, and conſequently DE^2 equals AH^2; ſubtract therefore equals from equals, that is, DE^2 from one ſide, and AH^2 from the

the other, and you will have $AF^2+DE^2+FC^2=2AHC+HC^2=2\,\overline{AH+HC}\times HC=2\,\overline{AH+2HB}\times HC=2AB\times HC=2AB\times 2BC=4ABC$; that is, $AF^2+DE^2+FC^2$ is equal to four times the rectangle ABC. Q. E. D.

PROBLEM 16.

324. *To find four lines in continual proportion, whereof both the sum of the extremes and the sum of the middle terms are given.*

This problem was proposed and solved algebraically in art. 164, putting a for the sum of the extremes, and b for the sum of the middle terms. There it was found, that the greater of the two middle terms was equal to $\frac{b}{2}\times\overline{1+\frac{\sqrt{a-b}}{\sqrt{a+3b}}}$, and the less equal to $\frac{b}{2}\times\overline{1-\frac{\sqrt{a-b}}{\sqrt{a+3b}}}$. Nothing here then remains to be done, but to furnish out a geometrical construction of this problem. But before that can be done, this quantity $\frac{\sqrt{a-b}}{\sqrt{a+3b}}$ must be more distinctly expressed thus: multiply both the numerator and denominator of that fraction by $\sqrt{a-b}$; and the numerator will then become $a-b$, and the denominator $\sqrt{a+3b}\times\sqrt{a-b}$. Find by the 13th of the sixth book of the Elements, a mean proportional between $a+3b$ and $a-b$, and call it m; and you will have $m^2=\overline{a+3b}\times\overline{a-b}$, and $m=\sqrt{a+3b}\times\sqrt{a-b}$; whence the foregoing fraction $\frac{\sqrt{a-b}}{\sqrt{a+3b}}$ will now become $\frac{a-b}{m}$: whence the greater of the two middle terms will be equal to $\frac{b}{2}\times\overline{1+\frac{a-b}{m}}$, and the less equal to $\frac{b}{2}\times\overline{1-\frac{a-b}{m}}$: and from these two expressions the following construction naturally flows. (*Fig.* 21.)

Draw the line BD equal to the given sum of the two middle terms; bisect it in H, and then from H towards B, or from H towards D set off HC, found by the 12th of the sixth book of the Elements thus: As a mean proportional between $a+3b$ and $a-b$ is to $a-b$ so make HB to HC, and the two segments BC and CD will be the two middle terms sought. Whence the extremes will easily be had thus: let the line BD be continued both ways, to wit, from B towards A, and from D towards E; then by the 11th of the sixth book of the Elements, make as BC to CD so CD to DE; make also as DC to CB so CB to BA, and

AB and DE will be the extremes ſought. So that the four lines ſought will be AB, BC, CD, DE. Q. E. I.

But here it muſt be obſerved, that b, the ſum of the middle terms, muſt always be leſs than a, the ſum of the extremes; for otherwiſe $a-b$ might be nothing or negative, in which caſes it would be impoſſible to find a mean proportional between $a+3b$ and $a-b$. This is alſo further evident from the 25th of the fifth book of the Elements, or by art. 291.

The compoſition of this problem has been ſufficiently demonſtrated in art. 164; and to give any other demonſtration of it here would create us more trouble than it is worth.

A LEMMA.

325. *In every rectilinear figure, all the angles taken together are equal to twice as many right ones, except four, as the figure has ſides.* Thus in a hexagon, where the number of ſides is 6, twice that number is 12, from which ſubtracting 4, there remains 8; therefore all the angles of any hexagon put together are equal to 8 right ones. Thus again, all the angles of a quadrilateral figure put together are equivalent to four right ones, &c.

This is the firſt of the two theorems uſually annexed to the 32d propoſition of the firſt book of the Elements. See *Barrow*, *Commandine*, *Clavius* and others.

PROBLEM 17. (*Fig.* 22.)

326. *In a given ſquare to inſcribe a leſſer ſquare of a given area.*

Let $ABCD$ be the given ſquare, upon whoſe ſides, beginning with AB, ſet off AE, BF, CG, DH all equal, and incloſe the figure $EFGH$. Now ſince in every quadrilateral figure all the angles taken together are equal to four right ones, as in the foregoing lemma, and ſince in the figure $EFGH$ no reaſon can be given why one angle ſhould be greater or leſs than another, or one ſide greater or leſs than another, it follows that all it's angles will be right ones, and all it's ſides equal, and conſequently that the figure $EFGH$ will be a ſquare inſcribed in the greater ſquare $ABCD$. What the problem requires then is to aſſign the point E in the ſide AB, and conſequently all the reſt, ſo that the ſquare $EFGH$ may have a given area.

SOLUTION.

Since the area of the ſquare $EFGH$ is given, the ſide EF will be given, that is, the hypotenuſe of the right-angled triangle EBF will be given, and we ſhall have $BF^2+BE^2=EF^2$, or $AE^2+EB^2=EF^2$: therefore we have now reduced this problem to the 13th in art. 321, which

which was, to divide a given line AB into two such parts AE and EB, that the sum of their squares may be equal to a given square EG.

N. B. From the limitation given to the 13th problem it appears, that the square EG must not be less than half the square AC.

PROBLEM 18.

327. *It is required, having given the difference between the hypotenuse of a right-angled triangle and each leg separately, to find the triangle.*

SOLUTION.

Let the two given differences be a and b, and put x for the hypotenuse; then will the legs be $x-a$ and $x-b$, and their squares $xx-2ax+aa$ and $xx-2bx+bb$, and the sum of their squares $2xx-2ax-2bx+aa+bb=xx$: make $a+b=c$, and you will have $aa+2ab+bb=cc$, and $aa+bb=cc-2ab$. Substitute therefore $cc-2ab$ instead of $aa+bb$ in the former equation, and also $-2cx$ instead of $-2ax-2bx$, and the equation will then be $2xx-2cx+cc-2ab=xx$, which being resolved gives $x=c\pm\sqrt{2ab}$, or $a+b\pm\sqrt{2ab}$: one of these roots, to wit $a-\sqrt{2ab}+b$ I reject, for a reason which will be given hereafter; the other root, to wit $a+\sqrt{2ab}+b$ I retain, which, considering that $\sqrt{2ab}$ is a mean proportional between a and $2b$, or between b and $2a$, gives the following construction. *(Fig. 23.)*

Draw the line AF; out of which take, first AB equal to either of the given differences, secondly BC equal to a mean proportional between that difference and twice the other, and thirdly CD equal to the other difference. This done, by the 22d of the first book of the Elements, set upon the base AD the triangle AED, making $AE=AC$, and $DE=DB$, and the triangle AED will be the triangle required.

A synthetical demonstration.

For 1st, that the lines AD, AC and DB are capable of constituting a triangle is plain, because every two of them are greater than the third: and 2dly, that the triangle AED is a right-angled triangle I thus demonstrate: by the 4th of the second book of the Elements, $AD^2=AC^2+2ACD+CD^2=AC^2+2AB\times CD+2BC\times CD+CD^2=AC^2+BC^2+2BCD+CD^2=AC^2+BD^2=AE^2+ED^2$: therefore the triangle AED is a right-angled triangle.

3dly, that the right-angled triangle AED is such a one as the problem requires is also evident; for the difference between AE and AD, or between AC and AD is CD, which was one of the differences given; and the difference between DE and DA, or DB and DA is AB, which was the other difference. *Q. E. D.*

SCHO-

SCHOLIUM.

In the foregoing solution we rejected one of the values of the hypotenuse, to wit, $a-\sqrt{2ab}+b$, and our reason for so doing is, that if the hypotenuse was made equal to $a-\sqrt{2ab}+b$, it would be impossible for both the legs to come out affirmative: for upon this supposition, one of the legs would be $a-\sqrt{2ab}$, and the other would be $b-\sqrt{2ab}$. Let us now suppose the leg $a-\sqrt{2ab}$ to be affirmative; then a must be greater than $\sqrt{2ab}$, and aa greater than $2ab$, and a must be greater than $2b$. In like manner, if we suppose the other leg $b-\sqrt{2ab}$ affirmative, b must be greater than $2a$: but it is impossible that both a should be greater than $2b$, and b greater than $2a$; therefore if the hypotenuse of the triangle be made equal to $a-\sqrt{2ab}+b$, it will be impossible for both the legs to be affirmative. *Q. E. D.*

If it be objected that negative legs of a right-angled triangle may be made affirmative without destroying the nature of such a triangle, it is allowed; but if our triangle should be so changed, it would not be such a one as the problem requires.

A LEMMA.

328. *If four right lines* A, B, C, D *be proportionable, and four others* E, F, G, H *be also proportionable, so that* A *is to* B *as* C *is to* D, *and also* E *is to* F *as* G *is to* H; *I say then that the rectangle* AE *will be to the rectangle* BF *as the rectangle* CG *is to the rectangle* DH.

For by the 23d of the sixth book of the Elements, the rectangle AE is to the rectangle BF in a ratio compounded of the ratio of A to B, and of the ratio of E to F. Again, the rectangle CG is to the rectangle DH in a ratio compounded of the ratio of C to D, and of the ratio of G to H: but the ratio of C to D is equal to the ratio of A to B by the supposition, and the ratio of G to H is equal to the ratio of E to F. Since then the component ratios are equal, the ratios compounded of them must be so too; that is, the ratio of the rectangle AE to the rectangle BF must be equal to the ratio of the rectangle CG to the rectangle DH.

Or thus: Since A is to B as C is to D, AE will be to BE as CG is to DG; and again, since E is to F as G is to H, BE will be to BF as DG is to DH: therefore the three rectangles AE, BE, BF are proportionable to the three rectangles CG, DG, DH; that is, AE is to BE as CG is to DG, and BE is to BF as DG to DH; therefore by the 22d of the fifth book of the Elements, or art. 287, AE is to BF as CG to DH. *Q. E. D.*

PRO-

PROBLEM 19.

329. *It is required, having given the hypotenuse of a right-angled triangle and it's area, to find the triangle.*

SOLUTION.

Call the given hypotenuse a, and by the 45th of the first book of the Elements find another line b of such a magnitude, that the rectangle ab may be equal to twice the area given; then will that rectangle ab also be equal to the rectangle of the two legs of the triangle sought; and therefore if x be put for one of the legs, $\frac{ab}{x}$ will be the other, and the sum of their squares will be $x^2+\frac{a^2b^2}{x^2}=a^2$; whence $x^4+a^2b^2=a^2x^2$, and $x^4=a^2x^2-a^2b^2$, and $x^4-a^2x^2 \quad -a^2b^2$, and $x^4-a^2x^2+\frac{a^4}{4}=\frac{a^4}{4}-a^2b^2$. Let c be a mean proportional between $\frac{a}{2}+b$ and $\frac{a}{2}-b$, that is, let $\frac{a^2}{4}-b^2=c^2$, and we shall have $\frac{a^4}{4}-a^2b^2=a^2c^2$, and the equation will now be $x^4-a^2x^2+\frac{a^4}{4}=a^2c^2$; whence by extraction of the square root, $x^2-\frac{aa}{2}=\pm ac$, and $x^2=\frac{a^2}{2}\pm ac=a\times\overline{\frac{a}{2}\pm c}$; therefore x is a mean proportional between a and $\frac{a}{2}\pm c$; that is, the greater leg of the triangle sought will be a mean proportional between a and $\frac{a}{2}+c$, and the less leg will be a mean proportional between a and $\frac{a}{2}-c$; whence we have the following construction. *(Fig. 24.)*

Draw the line AB equal to the hypotenuse given, and bisect it in C: then by the fortyfifth of the first book of the Elements, find a line CD such, that the rectangle $AB\times CD$ may be equal to twice the area given, and set off that line CD from C towards A, or from C towards B: find CE a mean proportional between AD and DB by the 13th of the sixth book of the Elements, and set that off also from C towards A or B. This done, by the twentysecond of the first book of the Elements describe the tri-

angle

angle AFB, so that AF may be a mean proportional between AB and AE, and BF a mean proportional between AB and BE, and the triangle AFB will be such a one as the problem requires.

This construction naturally flows from the foregoing *analysis*: for here $AB=a$, $AC=\frac{1}{2}a$, $CD=b$, $AD=\frac{1}{2}a\pm b$, $BD=\frac{1}{2}a\mp b$; therefore CE, which is a mean proportional between AD and DB, equals c; whence $AE=\frac{1}{2}a\pm c$, and $EB=\frac{1}{2}a\mp c$; therefore AF, which is a mean proportional between AB and AE, will be a mean proportional between a and $\frac{1}{2}a\pm c$; and for the same reason BF will be a mean proportional between a and $\frac{1}{2}a\mp c$. This construction then is what the foregoing *analysis* directly leads to: but by the demonstration annexed it will appear, that F the vertex of the triangle AFB will best be found by describing a semicircle upon the diameter AB which will meet the line EF perpendicular to AB in F the point required.

A synthetical demonstration.

1st. That the triangle AFB is right-angled at F is evident from it's being in a semicircle.

2dly. Therefore the triangles AEF and AFB are similar, and consequently AE will be to AF in one triangle, as AF is to AB in the other; that is, AF will be a mean proportional between AE and AB.

3dly. In like manner, from the similitude of the two triangles BEF and BFA, we shall have BE to BF as BF to BA.

4thly. Since then we have two sets of proportionals, to wit, AE to AF as AF to AB, and BE to BF as BF to BA, it follows from the last article, that the rectangle AEB will be to the rectangle AFB as that rectangle AFB is to the square of AB: but by the fifth of the second book of the Elements, the rectangle AEB with the square of CE is equal to the square of AC, and so also is the rectangle ADB with the square of CD; therefore $AEB+CE^2=ADB+CD^2$: subtract CE^2 from one side, and the rectangle ADB from the other, (for as CE is a mean proportional between AD and DB, CE^2 will be equal to $AD\times DB$,) and you will have the rectangle AEB equal to the square of CD: substitute therefore the square of CD instead of the rectangle AEB in the last proportion, and you will have CD^2 to $AF\times FB$ as $AF\times FB$ is to AB^2. Let m be a mean proportional between AF and FB, that is, let m^2 be equal to $AF\times FB$, and you will have CD^2 to mm as mm is to AB^2; therefore by the latter part of the 22d of the sixth book of the Elements, CD will be to m as m is to AB; therefore $mm=AB\times CD$: but $AB\times CD$ is double the given area of the triangle sought; therefore $AF\times FB$ is also double the area of the triangle sought; therefore AFB is the triangle sought. *Q. E. D.*

SCHOLIUM.

In the reſolution of moſt geometrical problems, after having any line as a given, it will be proper to refer to it as much as poſſible in all other expreſſions, rather than to introduce new lines and new expreſſions into the ſolution; eſpecially where the line to be referred is of any principal conſideration in the problem, or when the dimenſions run high. Thus in the foregoing problem, having the hypotenuſe a given, it was found more convenient to expreſs the double area of the triangle by the rectangle ab than by any other rectangle whatever, as bc, or even by any ſquare, as bb; the conſtruction derived from that *analyſis* being by this means rendered more ſimple than it would have been any other way, as will eaſily appear by comparing that *analyſis* with any other of the ſame problem.

Nor muſt I omit this opportunity of informing the reader, that though geometrical conſtructions drawn from analytical inveſtigations are (for the moſt part) ſureſt to be come at, yet they are not always moſt ſimple; for ſometimes it happens that the conditions of a problem are ſo independent one of another as to admit of diſtinct conſtructions, and that the *loci geometrici*, or geometrical places, (for ſo they are called,) whereby theſe diſtinct conſtructions are to be effected, are ſuch as offer themſelves naturally from the conditions of the problem without any algebraic computation. Whenever this happens, it will be eaſy, from the interſections of theſe *loci*, to form a conſtruction of the whole; and conſtructions thus formed will always be the ſimpleſt poſſible; a plain and eaſy inſtance whereof may be had from the problem now under conſideration. Let it again then be required, having given the hypotenuſe of a right-angled triangle, together with it's area, to find the triangle. *(Fig. 25.)*

Let AB be the given hypotenuſe; upon which, as on a *baſis*, deſcribe the right-angled parallelogram $ABCD$ equal to double the area of the triangle ſought: then by the 41ſt of the firſt book of the Elements it is evident, that every triangle having AB for it's baſe, and being terminated by the ſame parallels AB and CD, will be equal in area to the triangle ſought; and therefore the line CD is called the *locus* or place of the vertical angles of all ſuch triangles, as having AB for their common baſe, have the ſame area with the triangle ſought; therefore the vertex of the triangle ſought muſt be placed ſomewhere in the line CD; and ſo much for that condition. The other condition is, that the triangle ſought muſt be a right-angled triangle: therefore as all angles in a ſemicircle are right, it follows, that if upon the diameter AB be deſcribed a ſemicircle $AEFB$ on the ſame ſide with the parallelogram AC, and cutting the line CD in the points E and F, that arc $AEFB$ will

will be the *locus* of all the right angles that can possibly be subtended by AB; that is, the vertex of every right-angled triangle on that side, having AB for it's hypotenuse, will be in the arc $AEFB$; therefore the vertex of the triangle sought must have a place somewhere in the arc $AEFB$: but it cannot be both in the arc $AEFB$ and in the right line $DEFC$ unless it be in E or F one of the points of intersection; therefore AEB or AFB will be the triangle sought, those two triangles differing from one another in situation only.

If AD or BC, the breadth of the given parallelogram, be equal to $\frac{1}{2}AB$ or the *radius*, the points E and F now coinciding, the line CD will become a tangent to the middle of the arc, and the problem will be barely possible: but if the breadth of the parallelogram be greater than $\frac{1}{2}AB$, then the line CD and the arc can never meet, and the problem will become impossible when it's construction is so.

This is as clear an instance as can be given to illustrate the nature of these geometrical places we have above described: as to the rest we shall speak more hereafter.

A LEMMA.

330. *If four quantities* A, B, C *and* D *are proportionable, so that* A *is to* B *as* C *is to* D; *I say then that as* A+B *is to* A *so is* C+D *to* C.

For in the first place we have three quantities $A+B$, B, and A; in the next place we have three other quantities $C+D$, D, and C: of these it is plain that $A+B$ is to B in the first rank, as $C+D$ is to D in the second, by the 18th of the fifth book of the Elements, or art. 283: it is plain also that B is to A in the first rank, as D is to C in the second by the supposition and inversion; therefore by the 22d of the fifth book of the Elements, or art. 287, $A+B$ is to A as $C+D$ is to C. *Q. E. D.*

After the same manner might it be demonstrated that A—B *is to* A *as* C—D *to* C.

A LEMMA. (*Fig.* 26, 27.)

331. *If* ABC *be a triangle whose base is* AB, *and whose vertical angle* C *is equal to any given acute angle* DEF, *or to it's complement to two right ones; and if from any point as* D *in the line* ED *be drawn* DF *perpendicular to* EF; *I say then, that as* DE *is to* EF, *so will* 2ACB, *the double rectangle under the legs of the proposed triangle, be to* $AC^2+BC^2-AB^2$, *or to* $AB^2-AC^2-BC^2$, *the difference between the square of the base and the sum of the squares of the legs: the former case happens when the vertical angle* ACB *is equal to the acute angle* DEF, *and the latter happens when the vertical angle* ACB *is equal to the complement of the angle* DEF *to two right ones.*

I say likewise e converso, *that if* DEF *be any acute angle whatever, where* DF *is perpendicular to* EF, *and if* DE *be to* EF *as* 2ACB *is to* $AC^2+BC^2-AB^2$, *or to* $AB^2-AC^2-BC^2$; *then the vertical angle* ACB *will be equal to the angle* DEF *in the former case, or to it's complement to two right ones in the latter.*

For drawing the perpendicular AG to the side BC, or BC produced, the triangle ACG will be similar to the triangle DEF, the angles at F and G being right, and the angles DEF and ACG being equal by the supposition; therefore DE is to EF as AC is to CG: multiply the two last terms by $2BC$, and you will have DE to EF as $2ACB$ is to $2BCG$: but $2BCG$ is equal to $AC^2+BC^2-AB^2$, if the angle ACB be acute, or to $AB^2-AC^2-BC^2$, if the angle ACB be obtuse, by the 13th and 12th of the second book of the Elements; therefore in both cases we shall have DE to EF as the double rectangle of the legs AC and CB is to the difference between the square of the base and the sum of the squares of the legs.

It remains now that we demonstrate the converse of this proposition; to wit, that if DE be to EF as the double rectangle of the legs AC and CB is to the difference between the square of the base and the sum of the squares of the legs; then the vertical angle ACB will be equal to the angle DEF, or to it's complement to two right ones, according as that vertical angle happens to be acute or obtuse.

Since then by the supposition DE is to EF as the double rectangle of the legs AC and CB is to the difference between the square of the base and the sum of the squares of the legs; and since this last difference will always be equal to twice the rectangle BCG, whether the angle ACB be acute or obtuse, it follows that in both cases, DE will be to EF as $2ACB$ is to $2BCG$, that is, as AC is to CG: but if AC be to CG as DE to EF, since the angles at F and G are right, and consequently equal, the triangle ACG will be similar to the triangle DEF by the 7th of the sixth book of the Elements, and so the angle ACG is equal to the angle DEF: but the angle ACG is the vertical angle ACB, provided that vertical angle be acute, otherwise it is it's complement to two right ones; therefore the vertical angle ACB must, if acute, be equal to the angle DEF, and if obtuse, to the complement of that angle DEF to two right ones. *Q. E. D.*

PROBLEM 20.

332. *It is required, having given the base, the sum of the legs, and the angle opposite to the base of any triangle, to find the triangle.*

SOLU-

SOLUTION. (*Fig.* 28, 29.)

Let the vertical angle be equal to any given acute angle as DEF (*Fig.* 29,) where DF is perpendicular to EF, and call the ſum of the legs $2a$, the baſe $2b$, and the difference of the legs $2z$ (this way of notation being beſt for ſolving this problem,) and you will have the leſs leg equal to $a-z$, the greater equal to $a+z$, and the rectangle of the legs equal to a^2-z^2, and the double rectangle equal to $2a^2-2z^2$; you will alſo have the ſum of the ſquares of the legs equal to $2a^2+2z^2$, and the exceſs of this ſum above the ſquare of the baſe equal to $2a^2+2z^2-4b^2$: whence, and by the laſt article, we have this proportion, DE is to EF as $2a^2-2z^2$ is to $2a^2+2z^2-4b^2$; halve the two laſt terms, and you will have DE to EF as a^2-z^2 is to $a^2+z^2-2b^2$; therefore by art. 330, the ſum of the firſt and ſecond terms will be to the firſt as the ſum of the third and fourth is to the third, that is, $DE+EF$ is to DE as $2aa-2bb$ is to $aa-zz$; double the conſequents, and you will have $DE+EF$ to $2DE$ as $2aa-2bb$ is to $2aa-2zz$, or $DE+EF$ to $2DE$ as $aa-bb$ is to $aa-zz$.

On the center E and with the *radius* ED deſcribe the circle GDH, cutting the line EF both ways produced in G and H, to wit in G on the ſide of E, and in H on the ſide of F, and join GD, DH: then we ſhall have $GF=DE+EF$, and $GH=2DE$, and the proportion will now be that GF is to GH as $aa-bb$ is to $aa-zz$; multiply the two firſt terms by GH, and the rectangle FGH will be to GH^2 as $aa-bb$ is to $aa-zz$: but the triangle GDH is a right-angled triangle, as being in a ſemicircle, and will therefore be ſimilar to the triangle GFD; therefore FG will be to GD as GD is to GH; therefore the rectangle FGH will be equal to GD^2: ſubſtitute the latter inſtead of the former in the laſt proportion, and it will ſtand thus; GD^2 is to GH^2 as $aa-bb$ is to $aa-zz$. Take now AB (*Fig.* 28.) equal to $2b$ the given baſe of the triangle, and biſecting it in I, draw IN perpendicular to AB, and in the right angle AIN inſcribe the line AK equal to a the ſemiſum of the legs, and you will have $aa-bb=AK^2-AI^2=IK^2$, and the proportion will now ſtand thus; GD^2 is to GH^2 as IK^2 is to $aa-zz$. In the line IN take IL of ſuch a length, that IK may be to IL as GD to GH, and you will have IK^2 to IL^2 as GD^2 to GH^2: but as GD^2 is to GH^2 ſo was IK^2 to $aa-zz$; therefore IK^2 is to IL^2 as IK^2 is to $aa-zz$; therefore $aa-zz=IL^2$. In the angle AIL inſcribe the line LM equal to AK, and you will have $IL^2=LM^2-IM^2$; therefore $aa-zz=LM^2-IM^2$; but $a=LM$ by the conſtruction; therefore $z=IM$; therefore the difference of the two legs ſought will will be $2IM$. If therefore upon the baſe AB be

conſtructed

constructed a triangle ACB such, that the less leg AC shall be equal to $LM-MI$, and the greater leg BC shall be equal to $LM+MI$, the triangle ACB will be such a one as the problem requires. Q. E. I.

A synthetical demonstration.

Since $AC=LM-MI$, and $BC=LM+MI$, the sum of the legs AC and BC will be $2LM$ or $2AK$, as required: moreover, the rectangle of the two legs, to wit $\overline{LM-MI}\times\overline{LM+MI}$, will be $LM^2-MI^2=IL^2$, and the double rectangle of the legs will be $2IL^2$. Again, we have already shewn that the sum of the legs $AC+CB$ is equal to $2AK$, and therefore squaring both sides, we shall have $AC^2+2ACB+CB^2=4AK^2$; but $4AK^2$ is equal to $4AI^2+4IK^2$; therefore $AC^2+2ACB+CB^2=4AI^2+4IK^2$: subtract $2ACB$ from one side, and $2IL^2$ from the other, (since we have proved them equal,) and you will have $AC^2+BC^2=4AI^2+4IK^2-2IL^2$: subtract AB^2 from one side, and $4AI^2$ from the other, and you will have $AC^2+BC^2-AB^2=4IK^2-2IL^2$; that is, $4IK^2-2IL^2$ is the excess of the sum of the squares of the legs above the square of the base. This being allowed, the rest of the demonstration runs in a chain thus: GH is to GD as IL is to IK by the construction; therefore GH^2 is to GD^2 as IL^2 is to IK^2, or as $4IL^2$ to $4IK^2$: but it has been shewn already that GD^2 is equal to the rectangle FGH; therefore GH^2 is to GD^2 as GH^2 is to FGH, that is, as GH is to GF, or as $2DE$ is to $DE+EF$; therefore $2DE$ is to $DE+EF$ as $4IL^2$ is to $4IK^2$: halve the antecedents, and you will have DE to $DE+EF$ as $2IL^2$ is to $4IK^2$; therefore by the proportion of the 330th article inverted; the first term will be to the excess of the second above the first as the third is to the excess of the fourth above the third; that is, DE will be to EF as $2IL^2$ is to $4IK^2-2IL^2$: but $2IL^2$ is equal to $2ACB$, the double rectangle of the legs, as hath been shewn already, and $4IK^2-2IL^2$ was shewn to be the excess of the sum of the squares of the legs above the square of the base, to wit, $AC^2+BC^2-AB^2$; therefore DE is to EF as $2ACB$ is to $AC^2+BC^2-AB^2$; therefore by the last lemma, the vertical angle ACB is equal to the acute angle DEF. Q. E. D.

By a like process it will be found, that if the line IL be so taken as to have the same proportion to IK that the diameter GH hath to the lesser chord DH, the angle ACB will be obtuse, and equal to the angle DEG; but if the angle DEF or DEG be so given that IL happens to be greater than AK or LM, the problem will be impossible, because it will then be impossible to inscribe the line LM in the angle AIL: whence it follows, that if this problem be possible, the given vertical angle ACB must not be greater than AKB, whether that angle happens to be acute, right,

right, or obtuse; and that the triangle *AKB* will be the ultimate position of the triangle *ACB* before it becomes impossible.

SCHOLIUM. (*Fig.* 28, 29, 30.)

Let *AB* (*Fig.* 30) be the *basis* of the foregoing triangle, and bisecting it in *K*, draw the line *HIKL* perpendicular to *AB*; make the angle *BAI* equal to the angle *D* in the triangle *DEF* (*Fig.* 29,) and upon the center *I*, and with the *radius IA* or *IB*, describe the circle *AHBL*, which will be divided into two segments by the chord *AB*: I say then that the greater arc *AHB* will be the *locus* of all angles which standing upon *AB*, can be equal to the angle *DEF*; and that the lesser arc *ALB* will be the *locus* of all the angles which having the same subtense *AB* can be equal to the angle *DEG*.

For joining *AH*, *BH*; *AI*, *BI*; *AL*, *BL*, the triangle *AIK* will be similar to the triangle *DEF*, because the angles at *K* and *F* are right angles, and the angle *KAI* was taken equal to the angle *FDE* by construction; therefore the angle *AIK* is equal to the angle *DEF*: but the angle *AIK* at the center, standing but upon half an arc, is equal to the angle *AHB* at the circumference, standing upon the whole arc; therefore the angle *AHB* and consequently all others in that segment, by the 21st of the third book of the Elements, will be equal to the angle *DEF*; therefore all the angles *ALB* in the opposite segment will be equal to the angle *DEG*, by the 22d of the third book of the Elements; therefore the vertex *C* of the triangle *ACB* (*Fig.* 28,) must be somewhere in the arc *AHB* or *ALB* (*Fig.* 30,) according as the vertical angle *ACB* is acute or obtuse.

Again, there is scarce any one that is ever so little conversant in Conic Sections, but knows that if from any point, wherever it be taken in the perimeter of an *ellipsis*, be drawn two lines to the two *foci*, the sum of these two lines taken together will always be equal to the transverse *axis*: therefore, if upon the *foci A* and *B*, and upon a transverse *axis* equal to the sum of the legs given in the foregoing problem, be described an *ellipsis*, the perimeter of this *ellipsis* will be the *locus* of all points from whence lines drawn to *A* and *B* shall both together be equal to the sum of the legs given; therefore the vertex of the triangle sought must be somewhere in the perimeter of this *ellipsis*: but it was before shewn to be somewhere in the circumference of the circle above described; therefore either of the two points where either arc of the circle meets the *ellipsis*, if they do meet, will be the vertex sought; for each arc *AHB* or *ALB* may cut the *ellipsis* in two points. But if the arc wherein the vertex lies only touches the *ellipsis*, the two intersections will run into one, and so there will be but one point that can be made the vertex of the triangle sought. If the curves neither touch nor cut, the problem will be impossible.

This

This construction is obvious enough: but the laws of Geometry will never admit that a Conic Section be introduced into the construction of any problem which can be effected by the help of a right line and a circle: I say by the help of a right line and a circle; for inscribing the line *LM* in the angle *AIL* in the 28th figure, in order to find the point *M* on one side or other of the point *I*, was little less than describing a circle upon the center *L* and with a *radius* equal to *AK*, in order to find either of the intersections *M* wherein that circle would cut the line *AB*. Therefore whenever the conditions of a problem are constructed apart, if one of the *loci* happens to be a Conic Section, it will be most adviseable for the Analyst to suspend his acceptance of that construction, till (by extermination or otherwise) he has reduced the whole to one equation: then if that equation can be constructed by the help of a right line and a circle, or by the help of two circles, which is the case of all quadratics, and of all such as are reducible to quadratics, this last construction must be admitted rather than the former; but if it happens otherwise, the former construction may be allowed.

PROBLEM 21.

333. *It is required, having given two sides of any triangle, together with it's area, to find the third side.*

SOLUTION. (*Fig.* 31.)

The shortest and most direct way of solving this problem will be to construct the *locus* of each condition apart, that the intersection of these *loci* may point out the triangle sought. Take then the line *AB* equal to one of the given sides of the triangle, and to the line *AB* apply a right-angled parallelogram *ABCD* whose area shall be double of the given area of the triangle, by the 45th of the first book of the Elements. Again, upon the center *A*, and with a *radius* equal to the other given side, describe a circle cutting the line *CD* in *E* and *F*; then joining *AE*, *BE*; *AF*, *BF*, I say that either of the triangles *AEB* or *AFB* will answer the conditions of the problem, so that *BE* will be the third side of one triangle, and *BF* the third side of the other.

For by the 41st of the first book of the Elements, the area of each triangle will be half the area of the parallelogram *ABCD*, as it ought; and each triangle will have two sides, to wit, *AB* and *AE*, or *AB* and *AF* equal to the two sides given in the problem. *Q. E. D.*

If the third side *BE* or *BF* be required in numbers, imagine *EG* to be drawn perpendicular to *AB*: then in the right-angled triangle *AGE* the hypotenuse *AE* is known, and so also is the leg *EG*, because $\frac{1}{2}AB \times EG$,

$\times EG$, or the area of the triangle, was given in the problem; therefore AG will eaſily be obtained, and conſequently BG, ſince AB is known: ſince then in the right-angled triangle EGB, the legs EG and GB are known, the hypotenuſe EB will alſo be known, which is the third ſide of the triangle ſought.

A LEMMA. (*Fig.* 32.)

334. *If from two given points* A *and* B *be drawn two lines* AC *and* BD, *and if theſe lines, being infinitely continued beyond the points* C *and* D, *be ſuppoſed not to concur but at an infinite diſtance; I ſay then, that any finite parts as* AC *and* BD *of theſe infinite lines, ought to be taken for parallels, that is, they will have all the properties of parallel lines, ſo far as thoſe properties can be expreſſed in finite terms.*

For joining AB, draw at any diſtance the line CDE parallel to AB, and BE parallel to AC: and firſt let us ſuppoſe the lines AC and BD to meet in the point F, at a finite diſtance from AB, and the triangles EBD and AFB will be ſimilar; for the angle DBE will be equal to it's alternate AFB, and the angle BDE to it's alternate ABF; therefore DE will be to EB as AB is to AF: this is univerſally true, whatever be the diſtance of the interſection F from the line AB. Let us now ſuppoſe the point of interſection F to move off along the fixed line AF *ad infinitum*, and let the line BF be ſuppoſed to turn upon the fixed point B, ſo as always to paſs through the moveable point F, and ſuppoſing the figure $ACEB$ to continue as before, it is now evident that the line AB being finite will be infinitely leſs than the line AF which is infinite: but DE is to EB as AB to AF, and therefore in this caſe, the line DE is infinitely leſs than the finite line EB; therefore the line DE is leſs than can poſſibly be expreſſed in any finite terms, be they ever ſo ſmall. Since then the ſides BD and BE of the triangle BDE are finite lines, and the baſe DE is infinitely ſmall, the angle DBE muſt be leſs than any finite angle whatever, as eaſily follows from the 25th of the firſt book of the Elements: but the angle DBE is the difference of the two angles ABD and ABE; therefore the angle ABD, as far as it can be expreſſed in finite terms, is equal to the angle ABE: but the two angles ABE and BAC are equal to two right ones, becauſe by the ſuppoſition AC and BE are parallel; therefore the two angles ABD and BAC, as far as they can be expreſſed in finite terms, are equal to two right ones, and the lines AC and BD ought to be taken for parallel lines. *Q. E. D.*

Hence it follows, that all that can be demonſtrated concerning two lines AC and BD, with an infinitely ſmall allowance upon account of their concurrence at an infinite diſtance, will be ſtrictly and accurately

true of them without any allowance at all, when they become actually parallel. And thus may the properties of concurrent lines be carried on through this medium to parallels, an example whereof we have in the following problem.

PROBLEM 22.

335. *To divide a given triangle in any proportion by a line passing through a given point.*

CASE I. (*Fig.* 33.)

PREPARATION.

Let *ABC* be the triangle to be divided, and let *D* be the given point, and without the triangle, through which the dividing line is to pass. Now it is evident from a bare contemplation of the figures (33, 34, &c,) that in what situation soever the point *D* appears (either without the triangle, or in one of it's sides,) there can be but one angle of the triangle *ABC*, to which a line drawn from *D* shall pass through the triangle, either before it arrives at that angle, or afterwards when produced beyond it: let *C* be that angle, which we shall therefore call *the principal angle*, and let the line *DC*, when drawn, pass through the triangle before it arrives at the angle *C*, cutting the opposite side *AB* in *E*: divide also the side *AB* in *F*, so that the segments *AF* and *FB* may have the same proportion one to the other as the two parts into which the triangle *ABC* is to be divided, and join *CF*. Then it is plain that the two parts sought must be the same in quantity with the two parts *ACF* and *BCF*, since by the first of the sixth book of the Elements, the triangle *ACF* is to the triangle *BCF* as *AF* is to *FB*: therefore if the line *CF* when produced passes through *D*, the business is done, and the two parts sought will be the two triangles *ACF* and *BCF*. But if the line *CF* does not pass through *D*, as the line *CE* doth, then *AE* must either be greater or less than *AF*: let *AE* be greater than *AF*, and the triangle *ACE* will be greater than the triangle *ACF*, and consequently the other triangle *BCE* will be less than the triangle *BCF*: whence it follows, that to draw a line from *D*, so as to divide the triangle *ACB* into two parts equal to the parts *ACF* and *BCF*, the dividing line must decline from the line *DC* towards *A*, cutting off on the side of *A* a triangle equal to the triangle *ACF*, and consequently on the side of *B* a trapezium equal to the triangle *BCF*; and so *AB* and *AC* will be the two sides cut by the dividing line. After the same manner it may be demonstrated that if *AE* be less than *AF*, the dividing line must decline from *DC* towards *B*, cutting off on the side of *B* a triangle equal to the triangle *BCF*, and consequently on the side of *A* a tra-

a trapezium equal to the triangle ACF; in which case AB and BC will be the two sides cut by the dividing line. Let the former case obtain here, that is, let the sides AB and AC be cut by the dividing line, and for that reason be called *the principal sides:* draw DG parallel to AB, the side opposite to the principal angle, and meeting the other principal side CA produced in G, and the preparation will now be over, provided the problem is to be solved geometrically. But if the lines that are given, are given in numbers, and it be proposed to solve the problem arithmetically, the line AE, in order to know whether it be greater or less than AF, must be computed by the help of the similar triangles CGD and CAE, where CG is to GD as CA is to AE: but CA is given, because the triangle ABC is given, and CG and GD are given, because the situation of the point D with respect to the triangle is given; therefore the line AE is also giv

This preparation being made, the business of this problem will be to find a point as K in the line AC, through which and the point D a line as DHK being drawn, shall cut off a triangle as AHK equal to the triangle ACF.

SOLUTION.

Call AC a, AF b, AG c, DG d, and AK x; and since the triangle AHK is to be equal to the triangle ACF, they will have their sides about the common angle A reciprocally proportionable by the 15th of the sixth book of the Elements; that is, AK a side of the former triangle will be to AC a side of the latter, as AF the other side of the latter is back again, to AH the other side of the former; or according to our notation, x will be to a as b is to $\frac{ab}{x}$; therefore $AH=\frac{ab}{x}$. Again by similar triangles, GK will be to GD as AK to AH, that is, $x+c$ will be to d as x is to $\frac{dx}{x+c}$; therefore $AH=\frac{dx}{x+c}$. Thus we have two values of AH, to wit, $\frac{dx}{x+c}$ and $\frac{ab}{x}$, which therefore must be equal to one another, and give us the following equation, $\frac{dx}{x+c}=\frac{ab}{x}$; multiply both sides by x, and you will have $\frac{dxx}{x+c}=ab$; again, multiply by $x+c$, and you will have $dxx=abx+abc$, and $dxx-abx=abc$, and $xx-\frac{ab}{d}xx=\frac{abc}{d}$; to contract these expressions, make

 $\frac{ab}{d}$

$\frac{ab}{d}=e$, and $\frac{bc}{d}=f$, that is, make DG to AF as AC to e, and DG to AF as AG to f, and the equation will now ſtand thus, $xx-ex=af$; whence we ſhall have AK or $x=\frac{1}{2}e\pm\sqrt{\frac{1}{4}ee+af}$: but of theſe two roots or values of AK, one, to wit $\frac{1}{2}e-\sqrt{\frac{1}{4}ee+af}$ is negative; for if $\sqrt{\frac{1}{4}ee}$ be equal to $\frac{1}{2}e$, then $\sqrt{\frac{1}{4}ee+af}$ will be greater than $\frac{1}{2}e$, and this value of AK will be negative; that is, the point K found according to this value, will lie in the line CA produced beyond A, and conſequently on a contrary ſide of A with reſpect to the point C; and though the line DK, when drawn and produced through D till it cuts the ſide AB, will cut off an area equal in quantity to the area propoſed, yet lying without the triangle, it will not be the area propoſed; therefore there is but one value of AK that will ſolve the problem, to wit, $\frac{1}{2}e+\sqrt{\frac{1}{4}ee+af}$. Let g be the hypotenuſe of a right-angled triangle, one of whoſe legs is $\frac{1}{2}e$, and the other is $\sqrt{af}$, and you will have AK equal to $\frac{1}{2}e+g$. Join DA, and parallel to it draw FL meeting the ſide AC in L, and the triangles DGA and FAL will be ſimilar; for the angles DGA and FAL are equal, this laſt as being an external angle, and the former as being the internal and oppoſite angle on the ſame ſide; and the angles DAG and FLA will alſo be equal for the ſame reaſon; therefore DG is to AF as AG is to AL; but DG is to AF as AG is to f by the conſtruction; therefore $f=AL$, and $\sqrt{af}$ is a mean proportional between AC and AL; whence we have the following conſtruction of the problem.

Suppoſing all things as in the foregoing preparation, make DG to AF as AC is to e, and joining DA, draw FL parallel to it, and find g the hypotenuſe of a right-angled triangle one of whoſe legs is $\frac{1}{2}e$, and the other is a mean proportional between AC and AL. This done, if AK be taken equal to $\frac{1}{2}e+g$, and ſet off from A towards C, you will have the point K, through which and the point D the line DHK being drawn ſhall cut off the triangle AHK equal to the triangle ACF, and conſequently the trapezium $BHKC$ equal to the triangle BCF: and thus the triangle ABC will be divided in the proportion required by a line as HK, paſſing through the given point D. Q. E. F.

This might alſo be demonſtrated ſynthetically: but there are a great many other caſes belonging to this problem, as the reader will preſently ſee; and to give ſynthetical demonſtrations of them all, would be endleſs and tedious. I ſhall now therefore proceed to thoſe other caſes.

CASE 2. (*Fig.* 34.)

Let us then in the next place suppose the given point D to be in the side AB, and a more simple construction will arise; for then the line DG will coincide with the line DA, and the point G with the point A, in which case the line AG or c will vanish: therefore in this case, the quantity $\frac{bc}{d}$ or f, and the quantity $\frac{abc}{d}$ or af will vanish, and the algebraic canon for finding AK will now be reduced to this, *viz.* $AK = \frac{1}{2}e + \sqrt{\frac{1}{4}ee} = \frac{1}{2}e + \frac{1}{2}e = e = \frac{ab}{d}$: therefore when the point D lies in the side AB, the point K will be found only by drawing FK parallel to DC; for in this case AD will be to AF as AC to AK, and AK will be equal to $\frac{AC \times AF}{AD} = \frac{ab}{d}$. *Q. E. F.*

CASE 3. (*Fig.* 35.)

Let us now imagine the point D to pass through the side AB into the triangle, and all distinction of principal angles now ceasing, any two sides may be made principal sides; and the calculation will easily determine whether the supposition be just or not. But it will not be difficult for the artist himself to judge what lines are most proper for his purpose, as also which part to make the triangle, and which the trapezium, only by laying a ruler upon every angle, and upon the point D, and observing in what proportion it cuts the opposite sides. This (I say) will not be difficult, especially if he takes along with him the following consideration, which is in a manner self-evident, to wit, that if a ruler, as HDK be made to turn upon a fixed point D within a triangle, the area cut off by it, whether it be a triangle or a trapezium, will always increase on that side from which the longer part of the ruler is made to move. Thus if the part DK be longer than DH, the area of the triangle AHK will increase, if the ruler be made so to turn upon the point D as that the point K shall recede further and further from the point A, though at the same time the point H approaches nearer to it.

This premised, let AB and AC be the two sides cut by the line HDK as before; and drawing DG parallel to one of them, suppose to AB, cutting the other side AC in the point G, join DA, and draw FL parallel to it, cutting the side CA produced in L: and then it is plain that the point L having now changed sides with respect to the point A, the line AL or f must be reputed negative: the same is also further evident from the point G's having changed sides with respect to the same point

point A, upon which account the line AG or c will be negative; whence it follows that $\frac{bc}{d}$ or f, and $\frac{abc}{d}$ or af will be negative. Hence the algebraic canon for the resolution of this case will stand thus:

$$AK \text{ or } x = \tfrac{1}{2}e \pm \sqrt{\tfrac{1}{4}ee - af}.$$

I say $\pm$ because now both the values of AK will be affirmative, since $\sqrt{\tfrac{1}{4}ee - af}$ is less than $\tfrac{1}{2}e$. The construction therefore derived from this canon is as follows.

Let g be one leg of a right-angled triangle, whose other leg is a mean proportional between AC and AL considered as affirmative, and whose hypotenuse is $\tfrac{1}{2}e$; then setting off, from A towards C the line AK equal to $\tfrac{1}{2}e \pm g$, you will have the point K, through which the line KDH being drawn, will cut off the triangle AHK equal to the triangle ACF.

From this construction it follows, that if the values of AK be possible, the mean proportional between AC and AL must not be greater than $\tfrac{1}{2}e$; for that would be to suppose a right-angled triangle having one of the legs greater than the hypotenuse, which is absurd. But notwithstanding the foregoing construction, I must take notice to the reader, that there is a private article in this problem which neither was, nor could be included in the equation from which the foregoing canon and construction were derived; which is this. It is supposed in the problem that the line HK lies wholly within the triangle ABC, whereas that condition was not taken notice of in the equation: the equation was only founded upon this; that the area of the triangle AHK must be equal to that of the triangle ACF; which may happen even when AK comes out greater than AC, or AH greater than AB, though in neither of these cases will the line HK lie wholly within the triangle. In order therefore to judge, without the formality of a construction, whether the values of AK, and consequently of AH, be possible or not; and if possible, whether they will lie within proper limits, that is, whether AK will be less than AC, and AH be less than AB, take the following symptoms.

1st. *If the product* ab *be not less than the product* $4cd$, *the values of* AK *will be possible, otherwise not; and if the products be equal, the two values of* AK *will unite into one, as is usual in such cases.*

2dly. *If the two values of* AK *be possible, and* a *be less than* $e - f$, *only one of those values will be less than* AC.

3dly. *If* a *be greater than* $e - f$, *then both the values of* AK *will be less than* AC, *or neither of them; according as* a *is greater or less than* $\tfrac{1}{2}e$.

These rules being without much difficulty deduced from the canon above laid down, to wit, that AK is equal to $\tfrac{1}{2}e \pm \sqrt{\tfrac{1}{4}ee - af}$, I shall leave the

the investigation of them to the ingenious reader, rather than be thought to insist too long upon them. I shall only add, that after a proper value of AK is found, the line AH will easily be obtained, either by drawing the line KDH, or arithmetically by the proportion given in the resolution of the first case of this problem, to wit, that AK is to AC as AF to AH: and if, as AK is less than AC, AH be found less than AB, the problem will be resolved by making AB and AC principal sides; if not, other sides AB and BC, or AC and BC must be tried.

CASE 4. (*Fig.* 36.)

Let us now carry the point D into the angle C; and then the points C, D and K being united into one, if we make also the point H to coincide with the point F, the triangle AHK will be equal to the triangle AFC, and so the problem in this case will be resolved. But this is not all: my main design in this problem was to shew how from a canon appropriated to the first case only, all the other cases may be resolved, and to instruct the learner how to conduct himself through suchlike transitions. Therefore we do nothing, unless we demonstrate from the canon itself, that in this case AK is equal to AC, how evident soever it may be from other principles. Now in order to do this, we are to take notice that in this case, AG or c is equal to AC or a, and that DG or d is equal to nothing, as must always be the case where D lies in the line AC, or in AG produced: therefore in this case, $\frac{ab}{d}$ or e, and $\frac{bc}{d}$ or f must both be infinitely great, since their numerators are finite quantities, and their common denominator is infinitely small. It is plain also that these two quantities e and f will be to each other in a proportion of equality, since the two quantities a and c are so.

These things being observed, the canon belonging to the third case as it was derived from the first, stood thus: $AK = \frac{1}{2}e \pm \sqrt{\frac{1}{4}e^2 - af}$; or it may stand thus; $AK = \frac{1}{2}e \pm \sqrt{\frac{ee}{4} - \frac{aef}{e}}$. But here $\frac{f}{e}$ is equal to unity, it having been already shewn that e is to f in a ratio of equality: substituting therefore unity instead of $\frac{f}{e}$, the canon for the solution of the fourth case will be this:

$$AK = \frac{1}{2}e \pm \sqrt{\frac{1}{4}e^2 - ae}.$$

But this quantity $\frac{1}{4}e^2 - ae$ is equal to $e \times \overline{\frac{1}{4}e - a}$, and therefore must be infinite, since the factors are so; therefore $\sqrt{\frac{1}{4}e^2 - ae}$ must also be infi-

nite, and $\frac{1}{2}e+\sqrt{\frac{1}{4}e^2-ae}$ must be much more so: therefore in this fourth case, one of the values of AK must be infinite, and so can have no place in the resolution; but the other value, to wit $\frac{1}{2}e-\sqrt{\frac{1}{4}e^2-ae}$ may be finite for ought we know to the contrary: for though the sum of two infinite quantities must necessarily be infinite, yet their difference may be finite. This therefore is what now remains to be enquired into; and for our better success herein, we must first extract the root of $\frac{1}{4}e^2-ae$, as follows:

$$\frac{1}{4}ee-ae\left(\frac{1}{2}e-a-\frac{aa}{e}-\frac{2a^3}{e^2}\right.$$

$$\frac{1}{4}ee$$

$$\left.\begin{array}{l} e-a \\ \quad -a \end{array}\right)-ae \quad *$$

$$-ae \quad +aa$$

$$\left.\begin{array}{r} e-2a-\frac{aa}{e} \\ -\frac{aa}{e} \end{array}\right\}\begin{array}{l} -aa \quad * \quad * \\ -aa+\frac{2a^3}{e}+\frac{a^4}{e^2} \end{array}$$

$$\left.\begin{array}{r} e-2a-\frac{2a^2}{e}-\frac{2a^3}{e^2} \\ -\frac{2a^3}{e^2} \end{array}\right\}\begin{array}{l} -\frac{2a^3}{e}-\frac{a^4}{e^2} \quad * \quad * \\ -\frac{2a^3}{e}+\frac{4a^4}{e^2}+\frac{4a^5}{e^3}+\frac{4a^6}{e^4} \end{array}$$

Here then we see that $\sqrt{\frac{1}{4}e^2-ae}=\frac{1}{2}e-a-\frac{aa}{e}-\frac{2a^3}{e^2}$ &c *ad infinitum*. Now whoever considers this infinite series with any degree of attention, may easily see that it is of such a nature, that every subsequent term is infinitely less than that which goes immediately before it: for the first term $\frac{1}{2}e$ is infinitely great, and the second term a is a finite quantity; therefore the second term is infinitely less than the first. Again, the second term is a, and the third is $\frac{aa}{e}$; therefore the second term is to the third as a is to $\frac{a^2}{e}$, or as 1 is to $\frac{a}{e}$, or as e is to a: therefore the third term is infinitely less than the second. In like manner the third term is to the fourth as $\frac{a^2}{e}$ is to $\frac{2a^3}{e^2}$, or as e to $2a$: therefore the fourth term

term is infinitely less than the third; and so on *ad infinitum*. Whence it follows, that every term of this series, how small soever it may be, will be infinitely greater than all those that follow it put together: as if $-x$ be the sum of all the terms that follow the second $-a$, that second term $-a$ will be infinitely greater than $-x$, that is, an infinitely greater negative, or (which is the same thing) the quantity $-x$ will be infinitely small. Upon this supposition we shall have $\sqrt{\frac{1}{4}e^2 - ae} = \frac{1}{2}e - a - x$; therefore $\frac{1}{2}e - \sqrt{\frac{1}{4}e^2 - ae}$ or $AK = \frac{1}{2}e - \frac{1}{2}e + a + x = a + x = a = AC$, so far as it can be expressed in finite terms.

CASE 5. (*Fig.* 37.)

Suppose now the point D to pass again out of the triangle through the angle C; but let the line DC, when produced, pass into the triangle, so as to cut the line AB in E, and let AB and AC be determined principal sides, after the same manner as in the first case of this problem: draw DG parallel to AB, and meeting in G the side AC produced beyond C; and the point G having changed sides with respect to the point D, the line DG or d will now be negative, and therefore the line $\frac{ab}{d}$ or e will also be negative; but the line $\frac{bc}{d}$ or f will be affirmative, as in the first case; for the line c was negative before, and still continues to be so; and if the line d be negative too, the quantity $\frac{bc}{d}$ will have the same sign when c and d are both negative, as if they had been both affirmative. So the canon for this case will be as follows:

$$AK = -\tfrac{1}{2}e + \sqrt{\tfrac{1}{4}e^2 + af}.$$

Join AD, and draw FL parallel to it, meeting the line AC or AC produced in L, and let g be the hypotenuse of a right-angled triangle, one of whose legs is $\frac{1}{2}e$, and the other a mean proportional between AC and AL: then if from A towards C be set off the line AK equal to $g - \frac{1}{2}e$, the line DKH, when drawn, will solve the problem.

CASE 6. (*Fig.* 38.)

Lastly, from the angle C draw any line at pleasure within the triangle, as CE, cutting AB in E, and supposing the line EC to be infinitely continued beyond C, let us imagine the point D to move off to an infinite distance in the line EC produced; and by the foregoing lemma the line HK will at last become parallel to EC, these lines being now

ſuppoſed not to meet but at an infinite diſtance. Therefore the foregoing problem will now be changed into this, *viz*, *To divide a given triangle as* ABC *in any given proportion, ſuppoſe in the proportion of* AF *to* FB, *by a line as* HK *drawn parallel to any other line as* CE, *given in poſition*. To effect this it muſt be obſerved, that the line FL, which was always ſuppoſed parallel to AD, will now be parallel to EC, ſince by the lemma above-cited, any finite part of the line AD is parallel to EC; therefore the line AL or f will be finite and affirmative: but as the line DG or d is now infinitely great, the quantity $\frac{ab}{d}$ or e will be evaneſcent or nothing, having it's denominator infinitely greater than the numerator: therefore to adapt the canon of the firſt caſe to this, the quantities $\frac{1}{2}e$ and $\frac{1}{4}e^2$ muſt be ſtruck out of the canon, and then it will ſtand thus:

$$AK = \sqrt{af}.$$

Therefore if AK be taken a mean proportional between AC and AL, the line KH drawn parallel to CE will ſolve the problem. *Q. E. F.*

A diſcourſe concerning Infinites of both kinds, on occaſion of the foregoing problem.

336. By the reſolution of the foregoing problem it plainly appears, that infinitely great and infinitely ſmall quantities are not ſo very formidable in mathematical computations as they are generally taken to be, provided they be managed with judgement and diſcretion: nay ſo far are they from embarraſſing problems, that they frequently render their reſolution and conſtruction more eaſy and ſimple wherever they are concerned: and the reaſon is plain; for in all compariſon of finite quantities, one with another, infinitely ſmall ones are juſtly neglected, which, had they been finite, muſt have been taken notice of, and ſo have occaſioned more perplexity in the compariſon. And as to infinitely great quantities, theſe are either thrown out of the queſtion, as being improper for a finite ſolution, or elſe they paſs off by being converted into infinitely ſmall quantities. Thus the conſtruction of the 2d, 4th and 6th caſes of the foregoing problem became eaſier than the reſt, only on account of having infinitely ſmall quantities concerned in them: thus the infinite value of the quantity AK in the 4th caſe was thrown out of the queſtion, as not being wanted, and the infinite quantities e and f paſſed off in the infinitely ſmall ones $\frac{aa}{e}$, $\frac{2a^3}{e^2}$ &c.

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The true ſtate of the caſe is this. So long as we reaſon upon wrong ſuppoſitions, we muſt never expect to arrive at truth; but the nearer our ſuppoſitions are to truth, the nearer will be the concluſion; and if theſe ſuppoſitions be infinitely near the truth, the errors in the concluſion will be infinitely ſmall, which being at laſt thrown out of the account, the concluſion will be the ſame as if we had proceeded upon principles accurately true. This is the true riſe of infinitely ſmall quantities in all mathematical computations, and the true reaſon for rejecting them when the operation is over. But it may be reaſonably demanded, how do we know that theſe infinitely ſmall errors in the concluſion ariſe from like errors in the premiſſes? And the anſwer is, becauſe theſe two ſorts of errors have ſo mutual a dependence one upon another, that one cannot be made to vaniſh, but the other will neceſſarily vaniſh with it. If it be further demanded, what the wrong ſuppoſitions are from which theſe infinitely ſmall errors ſpring, I anſwer in the firſt place, the ſuppoſing magnitudes to have quantity which in reality have none at all, but have entirely loſt it, either by running into infinity on the one hand, or into nothing on the other. Thus in the reſolution of the 4th caſe of the foregoing problem, had we ſuppoſed in the reſolution, as we did in the caſe itſelf when it was firſt propoſed, that the point D actually coincided with the angular point C, (ſee *Fig.* 36,) the line DG or d would not only comparatively, but abſolutely have been equal to nothing; that is, it would have been no line at all, nor could any uſe have been made of it: therefore by allowing the line d to have ſome quantity, though infinitely ſmall, or leſs than any that can be aſſigned, we tacitly ſuppoſed the point D not to be actually in the angular point C, but infinitely near it. Hence aroſe the infinite quantities e and f, which therefore muſt not be ſuppoſed to be infinite or unbounded, as the word in the ſtricteſt ſenſe imports, but they muſt be looked upon as terminated lines, whoſe *termini* or extremes are at a greater than any aſſignable diſtance one from another: and theſe quantities being juſtly expunged at laſt in the infinitely ſmall quantities $\frac{aa}{e}$, $\frac{2a^3}{e^2}$ &c, we came to the ſame concluſion as if the ſuppoſition upon which it was founded had been accurately true. I ſaid that the quantities $\frac{a^2}{e}$, $\frac{2a^3}{e^2}$ &c were juſtly expunged, not only on account of their being infinitely ſmall, but alſo for the following reaſon: The line e in the reſolution was equal to $\frac{ab}{d}$; and therefore $\frac{aa}{e}$ was equal to $\frac{ad}{b}$; and for the ſame reaſon, $\frac{2a^3}{e^2}$ was equal to $\frac{2ad^2}{b^2}$: ſuppoſe now d,

the error in the supposition to vanish, and the infinitely small errors $\frac{ad}{b}$, $\frac{2ad^2}{b^2}$ &c will necessarily vanish with it. It is not to be denied, but that I might, in tenderness to the reader, have given the solution of this case a much easier turn; but to say the truth, I had a mind to shew him at once all he had to fear from these imaginary monsters, which (if he lays aside all unjust prejudices, and treats them as he ought) I can safely assure him he will find to be friends, and not enemies. But to proceed:

A quantity after it is reduced to nothing, ceases to be a quantity; and if o hath no quantity, neither can it's reciprocal $\frac{1}{o}$, or any magnitude expressed by it, be said to have any: but the reciprocal of nothing signifies a magnitude infinitely great in the strictest sense of the word, since as to it's infinity, no other can go beyond it: therefore a magnitude that is infinitely great in the strictest sense of the word, can no more be said to have quantity than absolute nothing can; and to compare such magnitudes in respect of their quantity, which actually have none, is contrary to the definition both of ratios and the object of proportion: nay I know not whether the greatest part, if not all the difficulties that are said to attend the idea of infinity, and our inability to comprehend it, ought not rather to be charged upon the absurdity of comparing things together which in their own natures are incapable of all comparison. It is said indeed that infinite parallelepipeds standing perpendicularly upon finite bases, and upon the same plane, are in proportion as their bases; which is true: but this is not comparing magnitudes in respect of the quantity they have not, but in respect of the quantity they have; one of these parallelepipeds may be said to be broader or thicker than another, though not higher. Thus if r be any quantity whereof the multiples $2r$ and $3r$ are taken, $2r$ may be said to be to $3r$ as 2 to 3, whether r be finite or infinite, or infinitely small; nay though r should signify an impossible quantity, as $\sqrt{-1}$, $\sqrt{-2}$, &c; but then the quantity of this proportion does not depend upon the quantity r, but upon the coefficients 2 and 3. But I must here take take notice however, that if r be actually infinite, I mean in the strictest sense of the word, by $2r$ and $3r$ must then be meant, not quantities twice or thrice as big as r, in the same respect wherein it is infinite, but r twice or thrice taken, which is no way absurd; for if it be possible for any one infinite quantity to exist that is not every way infinite, it will be as possible for others to exist of the same kind, independently of the former: a parallelepiped that is infinitely extended only as to it's length, and that both forwards and backwards, may however receive any addition, or be increased or diminished in any proportion in respect of it's finite dimensions, but not in respect to it's infinite extent; and this is all the propor-

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tion I can conceive infinite quantities capable of. See Philosophical Tranſactions, N°. 195.

But if we contract a little the ſenſe of the word, and by infinite quantities underſtand, not ſuch as are abſolutely boundleſs, but whoſe bounds are at a greater than any aſſignable diſtance one from another; and if, agreeably hereto, by an infinitely ſmall quantity we underſtand, not abſolute nothing, or a quantity whoſe bounds are coincident, but ſuch a one whoſe bounds are nearer to each other than any aſſignable diſtance; I muſt confeſs I do not ſee then why addition, ſubtraction, multiplication, diviſion, proportion, extraction of roots, figure, motion *&c*, ſhould not be as applicable to this ſort of quantities as to finite ones, and in the ſame ſenſe; nor can I ſee any abſurdity in this notion, ſince every quantity that is infinite in an abſolute ſenſe muſt neceſſarily include one that is infinite in this ſenſe. Let us aſſume any where a point, to which as to a center all others may be referred; and in this infinite expanſe of ſpace or matter, (for ſomething muſt be infinite,) I think it cannot be denied but that there actually exiſt parts of ſpace, and conſequently other points at a greater than any aſſignable diſtance from the point aſſumed; and if ſo, then the diſtance between any one of theſe points and the point aſſumed, will be a line whoſe extremes are nevertheleſs at a greater than any aſſignable diſtance one from another. After the ſame manner, by aſſuming lines or planes inſtead of points, may be demonſtrated the actual exiſtence of planes and ſolids whoſe *termini* are at a greater than any aſſignable diſtance from each other.

Whilſt I am upon this ſubject, it will perhaps be expected I ſhould ſay ſomething concerning the ultimate ratios of quantities increaſing and decreaſing *in infinitum*, ſo frequent in *Newton's* Philoſophy, whereof he was the firſt eſtabliſher. That great man made uſe of this doctrine in moſt of thoſe noble diſcoveries wherewith he has obliged the learned world; this way being much ſhorter than that which the ancients took in demonſtrating the truth of their propoſitions by ſhewing the abſurdity of the contrary, and more geometrical than the method of Indiviſibles ſo much cultivated by *Cavallerius*, *Torricellius* and others.

In order then to give the learner a proper idea of theſe ratios, we muſt firſt obſerve, that there are two ways whereby a quantity may paſs into nothing; either in time, when a quantity paſſes from it's preſent ſtate ſucceſſively through all leſſer degrees of magnitude into nothing, which the Mathematicians call vaniſhing; or in an inſtant, when a quantity paſſes at once from a finite ſtate into a ſtate of nothingneſs. As if a heavy body be thrown directly upwards with a certain degree of velocity, if it aſcends till by the continual force of it's gravity acting the contrary way, it has loſt all it's motion, this motion may be ſaid to be loſt in time: but if, during

during it's ascent, it happens to strike upon an obstacle above it, the motion at the time of the shock may be said to be lost in an instant; especially if both the body and the obstacle be supposed infinitely firm or hard. And the same ways that motion may be destroyed, it may be produced; for motion may be said either to be generated in time, as when a heavy body falls from rest, or to be communicated in an instant, as by the force of percussion.

N. B. By an instant is meant a point of time, and not a moment, or any small part of it.

Now to apply this distinction to our present purpose, let us suppose two variable quantities A and B to sink by degrees from their present state into nothing, so as to vanish both together; it is possible notwithstanding, that the ratio of A to B may all this while continue the same, or it may be constantly increasing, or constantly diminishing: but this is certain, that the very instant these quantities A and B lose their being, their ratio must do so too, though perhaps another way; for where there are no quantities, there can be no ratio. Let us then suppose (what is often the case) that at the very instant the quantities A and B vanish into nothing, their ratio passes at once from a finite state into nothing: this finite ratio, in the last instant of it's existence, is what is called *the ultimate ratio of the evanescent quantities* A *and* B. On the other hand if we suppose the quantities A and B to be generated from nothing, that is, to grow from nothing by degrees into a finite state; and if at the first instant of their existence we suppose their ratio to start into being the other way, that is, at once from nothing into a finite state, this finite ratio in the first instant of it's existence is called *the first ratio of the nascent quantities* A *and* B. Lastly, if we suppose the variable quantities A and B to pass from their present state successively through all greater degrees of magnitude into infinity, these quantities may lose their ratio this way as well as the other; and if the ratio breaks off whilst it is in a finite state, it is in that state said to be *the ultimate ratio of the quantities* A *and* B *increasing* ad infinitum.

N. B. By a finite ratio is meant the ratio that any one finite quantity may have to another.

One single example will be sufficient to illustrate the whole, which take as follows. Let x be any variable quantity, and make $4xx+3x=A$, and $2xx+x=B$; then will A and B also be variable quantities, as depending upon x; when x vanishes, A and B will both vanish, and when x is infinite, they will both be infinite: let it then be required to determine the ultimate ratio of A to B, whether they vanish into nothing or increase *ad infinitum*.

Before

Before I give a direct anſwer to this queſtion, give me leave to obſerve that the quantities A and B, whilſt they have any being, will be in a leſs ratio than that of 3 to 1, and in a greater than that of 2 to 1, which I thus demonſtrate: $6xx+3x$ is to $2xx+x$ as 3 to 1; (this is evident from the products of the extremes and means, which are equal:) but $4xx+3x$ is a leſs quantity than $6xx+3x$; therefore $4xx+3x$ is to $2xx+x$ in a leſs ratio than that of 3 to 1: but $4xx+3x=A$, and $2xx+x=B$ *ex hypotheſi*; therefore A is to B in a leſs ratio than that of 3 to 1. *Q. E. D.*

Again, $4xx+2x$ is to $2xx+x$ as 2 to 1; but $4xx+3x$ is a greater quantity than $4xx+2x$; therefore $4xx+3x$ is to $2xx+x$ (that is, A is to B) in a greater ratio than that of 2 to 1. *Q. E. D.*

But to come to the point: ſince $A=4xx+3x$, and $B=2xx+x$, A will be to B as $4xx+3x$ is to $2xx+x$; divide the two laſt quantities by x, and you will have A to B as $4x+3$ is to $2x+1$; therefore it is eaſy to ſee, that the leſs the quantities A and B are, that is, the nearer x is to nothing, the nearer will the ratio of $4x+3$ to $2x+1$ approach to the ratio of 3 to 1: when x is leſs than any aſſignable quantity, the ratio of $4x+3$ to $2x+1$, or the ratio of A to B, will approach nearer to the ratio of 3 to 1 than by any aſſignable difference; and when x vaniſhes, and conſequently $4x$ and $2x$ vaniſh with it, the ratio of A to B will terminate in the ratio of 3 to 1: therefore the ultimate ratio of the evaneſcent quantities A and B will be that of 3 to 1. *Q. E I.*

The ratio of 3 to 1 continues to be a ratio after the quantities A and B are vaniſhed, but it ceaſes then to be the ratio of A to B.

Again, ſince A is to B as $4x+3$ is to $2x+1$, (as was ſhewn before,) divide the two laſt quantities again by x, and you will have A to B as $4+\frac{3}{x}$ is to $2+\frac{1}{x}$. This is univerſal; but it eaſy to ſee that the greater the quantities A and B are, that is, the nearer the quantity x approaches to infinity, the leſs will be the quantities $\frac{3}{x}$ and $\frac{1}{x}$, and the nearer will the ratio of $4+\frac{3}{x}$ to $2+\frac{1}{x}$ approach to the ratio of 4 to 2 or 2 to 1. Let x be greater than any aſſignable quantity, and the quantities $\frac{3}{x}$ and $\frac{1}{x}$ will then be leſs than any that can be aſſigned; and the ratio of $4+\frac{3}{x}$ to $2+\frac{1}{x}$, or of A to B, will approach nearer to the ratio of 2 to 1 than by any aſſignable difference: therefore when x is in-

finite, and the quantities $\frac{3}{x}$ and $\frac{1}{x}$ are intirely vaniſhed, the ratio of A to B will terminate in the ratio of 2 to 1. Thus then we have found two limits between which the ratio of A to B will always conſiſt, to which it may, one way or other, approach nearer than any aſſignable diſtance, beyond which it cannot paſs, and at which it can never arrive, till the quantities whereof it is the ratio loſe their being, either in nothing or infinity: and as often as any limits of this kind are finite ratios, it muſt be the buſineſs of Mathematics to find them out.

At the beginning of my diſcourſe upon this ſubject I took notice, that infinitely ſmall errors were ſometimes unavoidable in computations founded upon wrong ſuppoſitions, and of theſe wrong ſuppoſitions I inſtanced in one only; but there are ſeveral others of the ſame ſtamp which Mathematicians are ſometimes obliged to make, and which at preſent I ſhall but juſt point at, having (I fear) been too prolix already: as when we ſuppoſe curve lines to conſiſt of an infinite number of infinitely ſmall ſtreight ones, or curvilinear areas to conſiſt of an infinite number of infinitely narrow parallelograms; when we ſuppoſe the curve ſurfaces of ſolids to be made up of an infinite number of infinitely narrow plain *annuli*, or their contents of an infinite number of infinitely thin *laminæ*; when we take the infinitely ſmall increments of quantities that do not flow uniformly, for their fluxions; when, in mechanical Philoſophy, we ſuppoſe a continued force to be made up of an infinite number of infinitely ſmall impulſes acting at infinitely ſmall intervals of time, and ſo forth: theſe ſuppoſitions (I ſay) are rather infinitely near the truth than accurately true; and therefore in our reaſoning upon them we muſt not be ſurprized if we ſometimes fall into infinitely ſmall errors, which muſt however be quaſhed before we can arrive at ſuch a concluſion as accurate reaſoning would naturally have led us to.

Mathematicians, and more eſpecially thoſe who beſt underſtand this ſubject, are, generally ſpeaking, reſerved enough upon it, and chuſe rather to be deficient than redundant in their expreſſions upon theſe occaſions; not from any diffidence in their own principles, but knowing very well how liable matters of this nature are to be drawn into diſputes by ſuch as lie upon the catch, and make it their chief buſineſs to oppoſe thoſe truths which they themſelves could never have diſcovered, nor perhaps will ever be able to underſtand. For my own part, it was not without a great deal of reluctance that I prevailed upon myſelf to ſay what I have done upon this head, conſidering how eaſily I might have avoided it: but knowing, by daily experience, how ſtrong a curioſity young Gentlemen have to pry into theſe notions, (perhaps more than they ought, unleſs they were better acquainted with all the other parts of mathematical

thematical knowledge,) and professing to write this treatise in favour of young beginners, who perhaps might not have it in their power to supply my defects, I thought it my duty to speak out upon this occasion, rather than leave them in worse hands: and if in so doing, particularly in my distinction betwixt absolute and comparative infinity, I have any way misrepresented the sense of the Mathematicians, I shall be very ready and willing to retract my error whenever I shall be apprized of it.

DEFINITIONS. (*Fig.* 39, 40.)

337. *Let* APB *be a line given in position, and let* PM *be an indeterminate line making any given angle* APM *with the line* AP: *let this indeterminate line* PM *be supposed to move along the line* AB *in a position always parallel to itself, so that all the* PMs *may be parallel to one another; and at the same time that the line* PM *is carried along the line* AP, *let the point* M *be also supposed to move along the line* PM, *so as by this compound motion to describe some streight or curve line: lastly, let there be a constant relation between the lines* AP *and* PM, *and let this relation be expressed by an equation involving those lines, or any powers of them multiplied or divided by known quantities: then is the streight or curve line described by the point* M *said to be the* locus *of that equation; the indeterminate line* PM *is called an ordinate; and the indeterminate line* AP, *comprehended between* P, *the foot of the ordinate, and the point* A, *which is always supposed to be a fixed point, is called the abscisse of the ordinate* PM. As for example, (*Fig.* 39,) call the abscisse AP, x, and the ordinate PM, y, and let p and q be any determinate lines, and let the constant relation between x and y be expressed by this equation $y = \frac{qx}{p}$: I say then that the *locus* of this equation will be a streight line, which I thus demonstrate.

Upon the line AP (produced if need be) set off from A towards P, the line AB equal to p, and draw BC equal to q, and parallel to PM, and on the same side of AP; then will the right line AC when drawn, be the *locus* of the foregoing equation. For if from any point as M in this line AC, be drawn MP parallel to BC, and if AP be called x, and PM y, the similar triangles ABC and APM will give AB (p) to BC (q) as AP (x) is to PM $(y) = \frac{qx}{p}$.

Again, (*Fig.* 40,) supposing the angle APM to be always a right one, and supposing r to be any given line, AP equal to x and PM equal to y as before, let the relation between x and y be expressed by this equation, $yy = rr - xx$: I say then that the *locus* of this equation will

 be

be the circumference of a circle whose center is A, and whose *radius* is equal to the line r. For if we suppose such a circle to be described, and if from any point of it, as M, we draw the line MP perpendicular to AP, and join AM, we shall have $AP^2 + PM^2 = AM^2$, that is, $xx + yy = rr$; whence $yy = rr - xx$. *Q. E. D.*

A LEMMA. (*Fig.* 41.)

338. *Supposing all things as in the last article, let the angle* APM *be always a right one, let* p, q *and* r *be given lines, and let the relation between* x *and* y *be expressed by this equation,* $xx - 2px + pp + yy - 2qy + qq = rr$. *Upon the line* AP *(produced if need be) and from* A *towards* P, *set off* AB *equal to* p, *and perpendicular to it draw* BC *equal to* q, *on the same side of* AB *with the line* PM: *I say then that the* locus *of the foregoing equation* $xx - 2px + pp + yy - 2qy + qq = rr$ *will be the circumference of a circle whose center is* C *and whose* radius *is* r.

For in the first place, $x^2 - 2px + p^2$ is the square of $x - p$ or of $p - x$, according as AP is greater or less than AB; that is, $x^2 - 2px + p^2$ will be the square of BP, whatever be the situation of the point P with respect to the point B; or if CD be drawn parallel to AP and meeting the line PM (produced if need be) in D, $xx - 2px + pp$ will be equal to the square of CD. Again, $yy - 2qy + qq$ is the square of $y - q$ or of $q - y$, according as PM is greater or less than PD, that is, $yy - 2qy + qq$ is the square of DM: substitute therefore in the foregoing equation CD^2 instead of $x^2 - 2px + p^2$, and DM^2 instead of $y^2 - 2qy + q^2$, and you will have $CD^2 + DM^2 = rr$: but by the 47th of the first book of the Elements, $CD^2 + DM^2 = CM^2$; therefore $CM^2 = rr$, and $CM = r$: therefore the point M must be somewhere and may be any where in the circumference of a circle whose center is C and whose *radius* is equal to the line r. *Q. E. D.*

N. B. 1st, If we have $+2px$ instead of $-2px$ in the foregoing equation, the line AB $(=p)$ must be set off the contrary way to AP; and if we have $+2qy$ instead of $-2qy$, the line BC $(=q)$ must be taken contrary to PM.

2dly, If the equation be $x^2 + y^2 \pm 2qy + q^2 = r^2$, that is, if the quantity p be wanting, it must be supposed equal to o; in which case (*Fig.* 42) the point B will coincide with the point A, and the center C must be found by drawing AC $(=q)$ perpendicular to AP, and on the same side with the line PM or contrarywise, according as $-2qy$ or $+2qy$ is found in the equation. Again, if q be wanting, the point C (*Fig.* 43) will coincide with the point B, that is, the center C will be somewhere in the line AP, if need be produced; and it will be found by setting

off AC $(=p)$ either from A towards P or the contrary way, according as $-2px$ or $+2px$ is found in the equation.

3dly, If the equation be $x^2 \pm 2px + p^2 + y^2 \pm 2qy + q^2 = -r^2$, the *radius* of the circle will be $\sqrt{-r^2}$; and as this is an impossible quantity, it shews there is no such line as MP that can have such relation to AP as is expressed in the equation. But if the equation be $x^2 \pm 2px + p^2 + y^2 \pm 2qy + q^2 = 0$, the *radius* of the circle will be 0, that is, the circle will now be contracted into one point C which before was it's center.

4thly, If the line AP be indefinitely produced both ways, it will pass through the circle or not, according as the *radius* r is greater or less than the line q.

5thly, If either of the quantities p^2 or q^2, or both be wanting in the equation, they must be supplied. As if the equation be $x^2 \pm 2px + y^2 \pm 2qy = \pm s^2$, by compleating the imperfect squares $x^2 \pm 2px$ and $y^2 \pm 2qy$, we shall shall have $x^2 \pm 2px + p^2 + y^2 \pm 2qy + q^2 = p^2 + q^2 \pm s^2$: make $p^2 + q^2 \pm s^2 = r^2$, and r will be the *radius* of the circle.

PROBLEM 23. (*Fig.* 44.)

339. *It is required, having given two points* A *and* B, *to find a third, as* M, *to which the lines* AM *and* BM *being drawn shall be in a given ratio.*

SOLUTION.

Let AM be to BM as a to b; and because we shall have occasion in the process for a third proportional to the two quantities a and b, let this third proportional be called c; let a be greater than b, and consequently b greater than c, that is, of the two points A and B, let that be marked with A which is most remote from the point M: join AB, and call it d, and draw MP perpendicular to it; and there will be three cases of this problem; for the point P may either fall between A and B, or upon B, or beyond it: let us suppose the first case, to wit, that P falls between A and B, and see whether a solution of this case will not take in all the rest.

Call then AP x, MP y, BP $d-x$, and you will have $AM^2 = x^2 + y^2$, $BM^2 = x^2 - 2dx + d^2 + y^2$; and therefore AM^2 will be to BM^2 as $x^2 + y^2$ is to $x^2 - 2dx + d^2 + y^2$: but AM is to BM as a to b by the supposition; therefore AM^2 is to BM^2 as a^2 is to b^2: whence we have the following proportion, $x^2 + y^2$ is to $x^2 - 2dx + d^2 + y^2$ as a^2 to b^2; multiply extremes and means, and you will have $aaxx - 2aadx + aadd + aayy = bbxx + bbyy$; transpose $bbxx + bbyy$, and you will have $aaxx - bbxx - 2aadx + aadd + aayy - bbyy = 0$; divide by $aa - bb$, and you will have

$xx\dfrac{-2a^2dx+a^2d^2}{a^2-b^2}+y^2=0$; transpose $\dfrac{a^2d^2}{a^2-b^2}$, and you will have $xx-\dfrac{2a^2dx}{a^2-b^2}+yy=-\dfrac{a^2d^2}{a^2-b^2}$: but a, b and c are continual proportionals by the supposition, and therefore $b^2=ac$ and $a^2-b^2=a^2-ac$: substitute therefore a^2-ac instead of a^2-b^2 in the foregoing equation, and reduce (as far as possible) the fractions to lower terms, and the equation will stand thus; $xx-\dfrac{2adx}{a-c}+y^2=-\dfrac{ad^2}{a-c}$. Now as this equation includes two unknown quantities x and y, it cannot determine either of them, but only shews the relation they have one to the other: therefore this problem admits of an infinite number of solutions; that is, there are an infinite number of points, as M, to which the lines AM and BM being drawn, the former will be to the latter as a to b; and by the last article it further appears, that the *locus* of all these points M will be the circumference of a circle; for the quantity $\dfrac{ad}{a-c}$ in this equation answers to the quantity p in the equation of the last article, and the quantity q in that equation is here equal to nothing; therefore by the second observation foregoing, the center of the circle will be somewhere in the line AB produced beyond B, and the distance of this center from the point A will be $\dfrac{ad}{a-c}$. I said that the center will be in the line AB produced, because the distance $\dfrac{ad}{a-c}$ is greater than $\dfrac{ad}{a}$ or d or AB.

We are in the next place to enquire into the length of the *radius*; and to do this we are to take notice, that the quantity pp in the foregoing equation, that is, $\dfrac{a^2d^2}{\overline{a-c}^2}$ is wanting; this therefore being added to both sides, the equation will be $xx-\dfrac{2adx}{a-c}+\dfrac{a^2d^2}{\overline{a-c}^2}+y^2=\dfrac{a^2d^2}{\overline{a-c}^2}-\dfrac{ad^2}{a-c}$; multiply both the numerator and denominator of this last fraction $\dfrac{[illegible]}{a-c}$ into $a-c$, that it may have the same denominator with the other that is joined with it, and the fraction will then be [illegible]; add this fraction to the other on the same side, that is, to the fraction $\dfrac{a^2d^2}{\overline{a-c}^2}$

losopher, and (I hope) to Philosophers, cannot think myself ill employed, or my reader ill entertained, whilst we are contemplating the nature of truth, and observing by what various shifts the consistency of things is maintained and preserved.

SCHOLIUM 2. (*Fig.* 46.)

Upon the solution of the foregoing problem depends that of the following. Let A, B and C be the centers of the bases of three columns standing upon the plane ABC, whose diameters let be a, b and c respectively: It is required to find a point, as M, in the plane of the triangle ABC, from whence these columns being viewed shall have their apparent diameters all equal.

The apparent diameter of a round object viewed at a distance, is the angle at the eye under which the true diameter is seen, and which is subtended by it: whence, reasoning as in art. 304, it will follow that the apparent diameter of any round object viewed at a distance will be as it's true diameter directly, and as it's distance from the eye reciprocally: therefore the apparent diameter of the column A, seen from the point M, will be to the apparent diameter of the column B, seen from the same point, as $\frac{a}{MA}$ is to $\frac{b}{MB}$: but this problem requires that these apparent diameters be equal; therefore the quantity $\frac{a}{MA}$ must be equal to the quantity $\frac{b}{MB}$, or to speak more properly, MA must be to MB as a to b: therefore if, according to the last problem, the circumference of a circle be described, the distance of every point whereof from A shall be to the distance of the same point from B as a to b, the point M must be somewhere in the circumference of this circle: in like manner, if the circumference of a circle be described, the distance of every point whereof from B shall be to the distance of the same point from C as b to c, the point M will be somewhere in the circumference of this circle also: therefore if these two circles intersect or touch each other, the point of view M will be in the points of intersection, or in the point of contact; otherwise the solution will be impossible.

If the diameters a, b and c be all equal, the point M (*Fig.* 47) will be in a perpendicular passing through the middle of AB, by the first scholium, and also in a perpendicular passing through the middle of BC, by the same scholium; and therefore it will be in the intersection of these two perpendiculars, that is, by the 5th proposition of the fourth book of the Elements, the point M will be in the center of a circle passing through

through the three points *A*, *B* and *C*, as it ought. And here it is plain there will be but one point of view: for had these *loci* been circles intersecting each other in two points, there would have been two points of view; but as in this case, the circles transform themselves into streight lines, there can be but one intersection, or one point of view; for the other must now be supposed to have gone off to an infinite distance.

Scholium 3.

By the three last articles the learner will be enabled to form to himself some sort of idea of what is meant by the *loci geometrici*, and of their use in the resolution and construction of geometrical problems; but it will be impossible for him to conceive thoroughly of these matters without a thorough knowledge of the Conic Sections, those lines being the *loci* generally made use of in the construction of all equations above the degree of quadratics: I shall therefore recommend to his perusal the late Marquiss *de L'Hospital*'s treatise of Conic Sections, which, though a posthumous piece, is nevertheless very correct, except in a few places; some whereof, I cannot but think, have been too severely animadverted upon of late; especially if it be considered how easily they are corrected, and without all doubt would have been corrected, had the author lived to put his last hand to this work. This treatise is, as I take it, by much the plainest, easiest, and best of it's kind I ever met with, for communicating a great deal of knowledge in a very little time; and it is upon this score chiefly that I venture to recommend it. In the four last books of this work we have a clear and full account of the *loci geometrici*, their construction and use, illustrated with a sufficient number of proper examples. In short, the whole design of this excellent performance is, in every respect, so very well laid, and carried on with so much facility, perspicuity and judgement, that it would be an inexcusable vanity in me so much as to imagine I could add any thing to so finished a piece. I shall therefore satisfy myself with having just given the learner a taste of these matters, and shall now proceed to other subjects.

BOOK

BOOK VIII. PART II.

Of Prisms, Cylinders, Pyramids, Cones and Spheres.

MANY of the following articles concerning the circle, sphere and cylinder are taken out of *Archimedes*, but demonstrated another way: and though they have no immediate relation to Algebra, yet as there are not many of them, and as they are a sort of supplement to *Euclid*'s Geometry, I have been prevailed upon to insert them here, for the sake of those who cannot read *Archimedes*, and for the ease of those who can. Moreover, as *Euclid*'s doctrine of solids is somewhat hard of digestion as it is delivered in the Elements, I have not scrupled to transfer some of the chief properties of cones and pyramids into this book, and to demonstrate them after a more easy and simple manner. And lastly, as the mensuration of the circle is absolutely necessary to the mensuration of the cylinder, cone and sphere, I shall, before I enter upon the rest, explain what *Archimedes* has delivered upon that head.

A Lemma.

340. *If in a right-angled triangle one of the acute angles be thirty degrees, or a third part of a right one, the opposite side will be equal to half the hypotenuse.* (Fig. 48.)

Let *ABC* be a right-angled triangle, right-angled at *B*, and let the angle *BAC* be 30 degrees; I say then that the opposite side *BC* will be half the hypotenuse *AC*.

For producing *CB* beyond *B* to *D*, so that *BD* may be equal to *BC*, and drawing *AD*, the two triangles *ABC* and *ABD* will be similar and equal; therefore the angle *CAD* will be 60 degrees, and the lines *AC* and *AD* will be equal; therefore the other two angles at *C* and *D* will be 60 degrees each, and the triangle *ACD* will be equilateral; therefore the line *BC*, which is the half of *CD*, will also be the half of *AC*. *Q. E. D.*

A Lemma. *(Fig. 49, 50.)*

341. *Let* ABC *be a right-angled triangle, right-angled at* B; *and supposing two similar and equilateral polygons, one to be circumscribed about a circle,*

circle, and the other to be inscribed in it, let the angle BAC *be equal to half the angle at the center subtended by a side of either polygon: I say then that* AB *will be to* BC *as the diameter of the circle to the side of the circumscribed polygon; and that* AC *will be to* BC *as the diameter of the circle is to the side of the inscribed polygon.*

Let D be the center of the circle, let EFG be a side of the circumscribed polygon, touching the circle in the point F, and let HIK be the side of a like polygon inscribed, and let HK and EG be supposed parallel, so as to subtend the same angle at the center. Draw the lines DHE, DIF, DKG; then will the three triangles ABC, DEF and DHI be similar, having the angles at B, F and I right, and the angle BAC being equal to the angle EDF by the supposition; therefore AB will be to BC as DF to EF, or as $2DF$ to EG, that is, as the diameter of the circle is to the side of the circumscribed polygon; and AC will be to BC as DH to HI, or as $2DH$ to HK, that is, as the diameter of the circle is to the side of the inscribed polygon. *Q. E. D.*

If the angle BAC *be a 48th part of a right one,* AB *will be to* BC *as the diameter of any circle is to the side of a regular polygon of 96 sides circumscribed about it, and* AC *will be to* BC *as the diameter is to the side of a like polygon inscribed.* For if the line HIK be the side of an inscribed regular polygon of 96 sides, the arc HFK will be a 96th part of the whole circumference, or a 24th part of a quadrant, and the arc HF a 48th part of a quadrant; whence the angle EDF or HDI will be a 48th part of a right angle.

A THEOREM.

342. *The circumference of every circle is somewhat more than three diameters.* (Fig. 51.)

Let AB be the side of a regular hexagon inscribed in a circle whose center is C, and draw AC and BC; then will the angle at C in the triangle ABC be 60 degrees, as containing a 6th part of the whole circumference; therefore since AC and BC are equal, the other two angles at A and B will be 60 degrees each; therefore the triangle ABC will be equiangular, and consequently equilateral; therefore AB will be equal to AC, and $6AB$ to $6AC$; but $6AB$ is equal to the perimeter of the inscribed hexagon, and $6AC$ is equal to three diameters; therefore the perimeter of a regular hexagon inscribed in a circle is equal to three times the diameter of that circle: whence it follows that the circumference of the circle itself will be somewhat more than three diameters. *Q. E. D.*

A THE-

A Theorem.

343. *If the diameter of a circle be called 1, the circumference will be somewhat less than* $3\frac{10}{70}$, *and somewhat greater than* $3\frac{10}{71}$.

The demonstration of the first part. (Fig. 52.)

Let *ABC* be a right angle, in which inscribe the lines *AC*, *AD*, *AE*, *AF*, *AG* in the manner following: make the angle *BAC* a third part of a right one, *BAD* a 6th part, *BAE* a 12th part, *BAF* a 24th part, and *BAG* a 48th part: then will *AC* be double of *BC* by the 340th article, and *AB* will be to *BG* as the diameter of any circle is to the side of a regular polygon of 96 sides circumscribed about it by the 341st article. Moreover as the line *AD* bisects the angle *BAC*, we shall have as *AB* to *AC* so *BD* to *DC* by the third of the sixth book of the Elements; and by art. 330, $AB+AC$ is to *AB* as *BC* is to *BD*; and by permutation, $AB+AC$ is to *BC* as *AB* is to *BD*: therefore if *BC* be divided into any number of equal parts, how many soever of these parts are contained in the sum of the lines *AB* and *AC*, the same number of like parts of *BD* will be contained in the line *AB* alone; as if *BC* be divided into 10000 equal parts, and the sum $AB+AC$ contains 37320 of those parts, then if the line *BD* be divided into 10000 equal parts, the line *AB* alone will contain 37320 of them. After the same manner it may be demonstrated, that whatever parts of *BD* are contained in the sum of the lines *AB*, *AD*, the same number of like parts of *BE* will be contained in *AB* alone, and so on: whence we have the following process.

1st. Let *BC* be divided into 10000 equal parts, or (which is the same thing) let *BC* be called 10000; then will *AC* be 20000, and consequently *AB* will be greater than 17320, and $AB+AC$ will be greater than 37320.

2dly. Therefore if $BD=10000$, *AB* will be greater than 37320, *AD* greater than 38636, and $AB+AD$ greater than 75956.

3dly. Therefore if $BE=10000$, *AB* will be greater than 75956, *AE* greater than 76611, and $AB+AE$ greater than 152567.

4thly. Therefore if $BF=10000$, *AB* will be greater than 152567, *AF* greater than 152894, and $AB+AF$ greater than 305461.

5thly. Therefore if $BG=10000$, *AB* will be greater than 305461; therefore *e converso*, if *AB* be supposed equal to 305461, *BG* will be less than 10000: but it was shewn before that *AB* is to *BG* as the diameter of any circle is to the side of a regular polygon of 96 sides circumscribed

cumscribed about that circle; therefore if the diameter of any circle be called 305461, the side of such a polygon will be less than 10000, and the whole perimeter less than 960000; therefore the perimeter of such a polygon will be less than the product of the diameter multiplied into $3\frac{10}{70}$ or $\frac{22}{7}$; for $305461 \times \frac{22}{7} = 960020\frac{2}{7}$: therefore if the diameter of any circle be called 1, the perimeter of a regular polygon of 96 sides circumscribed about it will be less than $3\frac{10}{70}$; but the circumference of every circle is less than the perimeter of any polygon circumscribed about it; therefore the circumference of the circle will still be less than $3\frac{10}{70}$. *Q. E. D.*

The demonstration of the second part. (Fig. 53.)

Let *ACDEFGB* be a semicircle whose diameter is *AB*, and in this semicircle let the lines *AC*, *AD*, *AE*, *AF*, *AG* be inscribed in the manner following: make the angle *BAC* a third part of a right one, *BAD* a sixth part, *BAE* a 12th part, *BAF* a 24th part, and *BAG* a 48th part, and join *BC*, *BD*, *BE*, *BF*, *BG*; then will *AB* be double of *BC*, and *AB* will be to *BG* as the diameter of any circle is to the side of a regular polygon of 96 sides inscribed. Let *AD* cut *BC* in *H*; and by the demonstration of the first part of this theorem, *AC*+*AB* will be to *CB* as *AC* to to *CH*, since by the construction the line *AH* bisects the angle *BAC*: but the triangles *ACH* and *ADB* are similar, having the angles at *C* and *D* right, as being in a semicircle, and the angle *CAH* being equal to the angle *DAB*; therefore *AC* will be to *HC* as *AD* to *BD*: but it was before demonstrated, that as *AC* is to *HC* so is *AB*+*AC* to *BC*; therefore as *AB*+*AC* is to *BC* so is *AD* to *BD*; and whatever parts of *BC* are contained in the sum of the lines *AB*, *AC*, the same number of like parts of *BD* will be contained in the line *AD* alone: whence the following process.

1st. Let *BC*=10000; then will *AB*=20000, *AC* will be less than 17321, and *AB*+*AC* will be less than 37321.

2dly. Therefore if *BD*=10000, *AD* will be less than 37321, *AB* will be less than 38638, and *AB*+*AD* will be less than 75959.

3dly. Therefore if *BE*=10000, *AE* will be less than 75959, *AB* will be less than 76615, and *AB*+*AE* will be less than 152574.

4thly. Therefore if *BF*=10000, *AF* will be less than 152574, *AB* will be less than 152902, and *AB*+*AF* will be less than 305476.

5thly. Therefore if *BG*=10000, *AG* will be less than 305476, and

and AB will be less than 305640; therefore *e converso*, if AB be equal to 305640, BG will be greater than 10000: but AB is to BG as the diameter of any circle is to the side of a regular polygon of 96 sides inscribed in it; therefore if the diameter of any circle be 305640, the side of such an inscribed polygon will be greater than 10000, and it's perimeter greater than 960000; therefore the perimeter of such a polygon will be greater than the product of the diameter multiplied into $3\frac{10}{71}$ or $\frac{223}{71}$; for $305640 \times \frac{223}{71} = 959968 - \frac{8}{71}$: therefore if the diameter of a circle be called 1, the perimeter of a regular hexagon of 96 sides inscribed in it will be greater than $3\frac{10}{71}$: but the circumference of every circle is greater than the perimeter of any inscribed polygon; therefore the circumference of this circle will be greater still than $3\frac{10}{71}$. *Q. E. D.*

Thus then if the diameter of a circle be called 1, the circumference must lie between these two very narrow limits, to wit $3\frac{10}{70}$ and $3\frac{10}{71}$: the whole difference of these limits is but $\frac{1}{497}$, and therefore by this method, the circumference of a circle is determined to a 497th part of the diameter.

The most compendious way of obtaining the numbers in the last article.

344. If any one has a mind to examine the foregoing calculations, it may not be amiss to let him know, that the hypotenuses of the triangles ABD, ABE, ABF and ABG (*Fig.* 52, 53) may be computed without either squaring the greater leg, or extracting the more considerable part of the square root. As if AD, (*Fig.* 52,) the hypotenuse of the triangle ABD in the first part be required, having given AB 37320 and BD 10000, the method I use is as follows.

1st, Whatever the hypotenuse AD may be, this is certain, that the greater leg AB will be equal to a considerable part of it; and therefore if AD be to be found by a series, as is usual in extracting the square root, it will be proper to make AB the first term; and hence it is that I call $37320 = AB$ my first root. Again, as the square of AD is to exceed the square of AB by the square of BD, that is, by 100000000: this number I call my first resolvend, and then doubling my first root, the

the product 74640 I call my first divisor, and so am prepared for the following operation.

2dly, Thus prepared, I divide my first resolvend by my first divisor, and the first figure of the quotient (for I am concerned for no more at present) I find to be 1, which, as it comes out of the place of thousands, signifies 1000; this number therefore 1000 I add to my first root, and so have 38320 for a more correct or second root. The same number 1000 I add also to my first divisor, and then multiplying the sum 75640 by 1000, the number that was added, I subtract the product 75640000 from my first resolvend, and there remains 24360000; this I call my second resolvend, and the double of my second root, to wit 76640, I call my second divisor, and so proceed to the next operation.

3dly, Now I divide my second resolvend by my second divisor, and the first figure of the quotient is 3, which, as it comes out of the place of hundreds, signifies 300; therefore I add 300 to my second root, and so have 38620 for my third root: the same number 300 I also add to my second divisor, and the sum 76940 I multiply by 300, and the product is 23082000, which being subtracted from my second resolvend, leaves me 1278000 for a third resolvend, and the double of my third root, to wit 77240, I have for my third divisor.

4thly, I divide my third resolvend by my third divisor, and the first figure of the quotient is 1, which signifies 10; therefore I add 10 to my third root, and so have a fourth root 38630: moreover adding 10 to my third divisor, the sum is 77250, which being multiplied by 10, and the product 772500 being subtracted from the third resolvend, leaves 505500 for the fourth resolvend, and the double of my fourth root, to wit 77260, makes a fourth divisor.

5thly and lastly, I divide my fourth resolvend by my fourth divisor, and the nearest quotient too little is 6; therefore I add 6 to my fourth root, and so have a fifth root, to wit, 38636, which is the nearest root less than the true that can be expressed in whole numbers: therefore the hypotenuse *AD* is greater than 38636.

The reason of these operations will not be difficult to any one who thoroughly understands the foundation of the common method of extracting the square root.

Van Ceulen's *numbers expressing the circumference of a circle whose diameter is* 1.

345. This method of *Archimedes* is capable of being pursued to any degree of exactness required: nay *Ludolf Van Ceulen* has computed the circumference of a circle to no fewer than 36 places, upon a supposition that

that the diameter is unity. His numbers expressing this circumference are

3 .1415 9265 3589 7932 3846 2643 3832 7950 288+.

But since the invention of fluxions by it's great author Sir *Isaac Newton*, he (Sir *Isaac*) has from this method drawn serieses almost infinitely more expeditious than the bisections of *Archimedes* or *Van Ceulen*, whereby the circumference of a circle may be computed to 12 or 13 places in little more than half an hour's time, as Doctor *Halley* from his own experience assures us.

Note, that *Metius*'s proportion of the diameter of a circle to the circumference is as 113 to 355, the most accurate of any in such small numbers. (See Schol. 1. in art. 179.)

Why the circle cannot be squared geometrically.

346. If, having given the diameter or semidiameter of any circle, a right line could be found exactly equal to the circumference, whether such a line could be expressed by numbers or not, the circle might be squared as well as any right lined figure whatever, that is, a square might be constructed whose area would be equal to that of the circle, which I thus demonstrate.

Let $2r$ represent the diameter of any circle, and $2c$ the circumference; then will rc, the product of the *radius* into the semicircumference be it's area, by cor. 4 in art. 311. Let now x be the side of a square whose area is equal to that of the circle, and we shall have $xx = rc$; whence x will be a mean proportional between r and c, and may be found by the 13th of the sixth book of the Elements. But it is impossible upon *Euclid*'s *postulata*, having given the diameter or semidiameter of any circle to find a right line exactly equal to the circumference; and therefore the circle cannot be squared upon the same foundation on which we are taught to square all right-lined figures; and hence it is that we say, the circle cannot be squared geometrically. But as the three *postulata* of *Euclid*, how simple soever they may appear in theory, are never a one of them capable of being perfectly executed, but that errors must necessarily arise in drawing and producing lines, in taking the distances of points, &c; and as from these errors others must necessarily arise in subsequent operations; and lastly, as the circumference of a circle may be had from the diameter in numbers, to any assignable degree of exactness, it follows that in practice, a circle is as capable of being squared as any other figure whatever that is not a square.

Corollaries drawn from art. 343.

347. From the 343d article may be deduced several corollaries, some of the most useful whereof are inserted here as follows.

1st. *The diameter of every circle is to the circumference as 7 to 22 nearly:* for 1 is to $3\frac{10}{70}$ or $\frac{22}{7}$ as 7 to 22.

2d. *If* d *be the diameter of any circle, it's area will be* $\frac{11dd}{14}$: for as 7 is to 22, so is d the diameter to $\frac{22d}{7}$ the circumference; and if $\frac{d}{2}$ the *radius* be multiplied into $\frac{11d}{7}$ the semicircumference, the product $\frac{11dd}{14}$ will be the area, by corollary 4 in art. 311.

3d. *Hence we have a ready way, having the diameter of any circle given to find it's area, and* vice versa, *without the mediation of the circumference, by saying, as* 14 *is to* 11, *so is the square of the given diameter to the area sought. But if the area be given, in order to find the diameter, the proportion must be reversed, by saying as* 11 *is to* 14; *so is the given area to a fourth, which fourth number will be the square of the diameter, and it's square root the diameter itself.*

4th. Arguing as in the two last corollaries, *If the diameter of a circle be to the circumference as* a *to* b, *then the square of the diameter of any circle will be to it's area as* 4a *to* b; *and* vice versa, *the area will be to the square of the diameter as* b *to* 4a.

5th. *The circumferences of all circles are as their diameters or semidiameters, and their areas as the squares of the diameters or semidiameters.* For if d and e be the diameters of two circles, their circumferences will be $\frac{22d}{7}$ and $\frac{22e}{7}$; and $\frac{22d}{7}$ is to $\frac{22e}{7}$ (dropping the common denominator 7, and the common factor 22) as d to e. Again, the area of the circle whose diameter is d is $\frac{11dd}{14}$ as in the second corollary; and for the same reason, the area of the other circle whose diameter is e is $\frac{11ee}{14}$; and $\frac{11dd}{14}$ is to $\frac{11ee}{14}$ as dd to ee; therefore the circumferences of all circles are as their diameters, and their areas as the squares of their diameters. And since

ſince the halves of all quantities are as the wholes, and the ſquares of the halves as the ſquares of the wholes, it follows alſo that the circumferences of circles are as their ſemidiameters, and their areas as the ſquares of the ſemidiameters.

6th. *If there be three circles ſuch, that the ſum of the ſquares of the ſemidiameters of two of them is equal to the ſquare of the ſemidiameter of the third; I ſay then that the areas of the two firſt circles put together will be equal to the area of the third.* For let a, b, c repreſent the ſemidiameters of the three circles, and let $a^2+b^2=c^2$: ſince then the ſemidiameter of the firſt circle is a, the diameter will be $2a$, and the ſquare of the diameter $4aa$: but as 14 is to 11 ſo is $4aa$ to $\frac{44a^2}{14}$ or $\frac{22a^2}{7}$; therefore the area of the firſt circle will be $\frac{22a^2}{7}$ by the third corollary; and for the ſame reaſon, the areas of the other two circles will be $\frac{22b^2}{7}$ and $\frac{22c^2}{7}$: but $a^2+b^2=c^2$ *ex hypotheſi*; therefore $\frac{22a^2}{7}+\frac{22b^2}{7}=\frac{22c^2}{7}$.

N. B. This laſt corollary is not demonſtrated in the 31ſt of the ſixth book of the Elements, as ſome may imagine, that demonſtration not reaching further than right-lined figures.

The following eaſy problems may ſerve to ſhew ſome uſes of the foregoing corollaries.

PROBLEM I.

348. *To find the proportion between the diameter of any circle and the ſide of an equal ſquare.*

Call this diameter 1, and by the ſecond corollary in the foregoing article, the area of this circle will be $\frac{11}{14}$ nearly; and the ſide of a ſquare whoſe area is $\frac{11}{14}$ will be $\sqrt{\frac{11}{14}}$: therefore the diameter of any circle is to the ſide of an equal ſquare as 1 to $\sqrt{\frac{11}{14}}$. But becauſe this fraction $\frac{11}{14}$, though it ſerves well enough for common purpoſes, is not accurately true, and becauſe it's ſquare root cannot be accurately expreſſed in numbers neither, to reduce the error (for there muſt be an error) to a more ſimple point, let c be the circumference of a circle whoſe diameter is 1; and the area of ſuch a circle, that is, the product of the

radius into the femicircumference will be $\frac{1}{2} \times \frac{c}{2} = \frac{c}{4}$; and the fide of an equal fquare will be $\sqrt{\frac{c}{4}}$: but according to *Van Ceulen*, $c = 3.1415926536$, and $\frac{c}{4} = .7853981634$, and $\sqrt{\frac{c}{4}} = .88623$; therefore the diameter of a circle is to the fide of an equal fquare as 1 to .88623, or as 100000 is to 88623: fuppofe the proportion to be that of 100000 to 88625, which makes but an error of 2 in the fifth place, and then multiplying both terms by 8, the proportion will be that of 800000 to 709000, or of 800 to 709; therefore *As* 800 *is to* 709, *fo is the diameter of any circle to the fide of an equal fquare, nearly true to five places.*

N. B. If the diameter of a circle be 9 yards, the fide of an equal fquare found as above, will not err an hundredth part of an inch.

PROBLEM 2.

349. *To find the femidiameter of a circle that will comprehend within it's circumference the quantity of an acre of land.*

An acre of land contains 4840 fquare yards, and therefore this muft be the area of our circle. Say then as 11 to 14 fo 4840 to 6160; and this laft number will be the fquare of the diameter, by the third corollary in art. 347; whence the diameter itfelf will be 78.486 yards, and the femidiameter 39.243 yards, that is 39 yards $8\frac{3}{4}$ inches nearly.

PROBLEM 3.

350. *Let a ftring of a given length be difpofed into the form of a circle: It is required to find the area of this circle.*

Let c be the length of the ftring, and confequently the circumference of the circle made by it, and the diameter of the circle will be $\frac{7c}{22}$, the femidiameter $\frac{7c}{44}$, and the area $\frac{7cc}{88}$. Suppofe now this ftring to be difpofed into the form of a fquare, and the fide of this fquare would be $\frac{c}{4}$, and it's area $\frac{cc}{16}$, and the area of the fquare would be to the area of the circle as $\frac{cc}{16}$ is to $\frac{7cc}{88}$, that is, as $\frac{1}{16}$ is to $\frac{7}{88}$, or as 11 to 14: therefore *As* 11 *is to* 14, *fo is the area comprehended by the ftring when in form of a fquare, to the area comprehended by the fame ftring when in form of a circle.* Q. E. I.

N. B.

N. B. By the anſwer here given it appears, that if c be the circumference of any circle, it's area will be $\frac{7cc}{88}$; and conſequently that *As* 88 *is to* 7, *ſo is the ſquare of the circumference of any circle to it's area nearly.*

PROBLEM 4.

351. *It is required to divide a given circle into any number of equal parts by means of concentric circles drawn within it.* (Fig. 54.)

Let A be the center, and AF be the ſemidiameter of the circle given, and let it be required to divide the area of this circle into five equal parts by means of four concentric circles deſcribed within the former, and cutting the line AF in the points B, C, D, E, that is, let the area of the circle AB, and the areas of the *annuli* BC, CD, DE, and EF be ſuppoſed all equal; then the circle AB will be $\frac{1}{5}$ of the whole, the circle AC $\frac{2}{5}$, the circle AD $\frac{3}{5}$ &c; and the area of the circle AF will be to the area of the circle AB as 5 to 1: but the area of the circle AF is to the area of the circle AB as the ſquare of the ſemidiameter AF is to the ſquare of the ſemidiameter AB, by cor. 5 in art. 347; therefore AF^2 is to AB^2 as 5 to 1: but AF^2 is given by the ſuppoſition; therefore AB^2, and conſequently AB the ſemidiameter of the inmoſt circle is given. In like manner AF^2 is to AC^2 as 5 to 2; whence AC the ſemidiameter of the next concentric circle is given; and ſo of the reſt. *Q. E. I.*

PROBLEM 5.

352. *Whoever makes a tour round the earth muſt neceſſarily take a larger compaſs with his head than with his feet: The queſtion is, how much larger?*

Let A *(Fig. 55.)* repreſent the center of the earth, AB it's ſemidiameter, BC the traveller's height, AC the the ſemidiameter of the circle deſcribed by his head: let alſo b repreſent the circumference of the circle whoſe ſemidiameter is AB, and c the circumference of the circle whoſe ſemidiameter is AC, and $c-b$ will be the difference we are now enquiring into, which may be thus determined.

By the fifth corollary in art. 347, AC is to AB as c is to b; and by diviſion of proportion, BC is to AB as $c-b$ is to b; and alternately, BC is to $c-b$ as AB is to b. Let d be the circumference of a circle whoſe ſemidiameter is BC, and BC will be to d alſo as AB to b; therefore BC is to d as BC is to $c-b$; therefore $c-b=d$; that is, *The traveller's head will paſs through more ſpace than his feet by the circumference of a circle whoſe ſemidiameter is his own length:* as if the man be 6

 feet

feet high, his head will travel further than his heels by 37 feet $8\frac{4}{7}$ inches nearly, and that whether the semidiameter AB be greater or less, or nothing at all.

PROBLEM 6.

353. *It is required, having given the depth and the diameter of the base of any cylindrical vessel, to find it's content in ale gallons.*

Here it must be observed, that in the mensuration of a solid, all it's dimensions must be taken in the same kind of measure: as if any one dimension be taken in inches, the rest must be taken so too, and then the number representing the content of any solid will be the number of cubic inches to which that solid is equivalent.

Let then a be the given altitude of the cylindrical vessel to be measured, d the diameter of it's base, and by the second corollary in art. 347, $\frac{11dd}{14}$ will give a number of square inches equal to the base, and this area multiplied into the altitude a, will give $\frac{11add}{14}$, a number of cubic inches equal to the solid content of the vessel: but 282 cubic inches constitute an ale gallon; and therefore if $\frac{11add}{14}$, the solid content of the vessel, be divided by 282, the quotient, to wit $\frac{11add}{3948}$, will be the number of gallons therein contained. Instead of 3948 put 3949, which will make no considerable difference in so great a denominator, and the fraction $\frac{11add}{3949}$ (dividing both the numerator and denominator by 11) will be reduced to $\frac{add}{359}$: whence the following canon.

Having taken both the depth, and the diameter of the base in inches, multiply the square of the diameter into the depth of the vessel, and the product divided by 359 will give the number of gallons required.

N. B. This substitution of 3949 instead of 3948 corrects about $\frac{7}{10}$ of the error that would otherwise have been committed in supposing the square of the diameter of the base to be to it's area as 14 to 11.

PROBLEM 7.

354. *To measure a frustum of a cone, whose perpendicular altitude and the diameters of the two bases are given.*

N. B.

N. B. By a frustum of a cone I mean a cone having it's top cut off by a plane parallel to the base.

Let the isosceles triangle *ABC* (*Fig.* 56) represent the section of a cone made through it's *axis* *AD*, and let *EF* be any line parallel to the base *BC*, cutting *AB* in *E*, *AC* in *F*, and the *axis* *AD* in *d*; then will the trapezium *BEFC* be the section of a frustum of this cone whose perpendicular altitude is *Dd*. Call *BC*, the diameter of the greater base, g; *EF*, the diameter of the lesser base, l; and *Dd*, the height of the frustum, h: call also *AD*, the unknown altitude of the whole cone, x; and consequently *Ad*, the altitude of the cone to be cut off, $x-h$; and from the similar triangles *ABC*, *AEF* we have this proportion; *AD* is to *Ad* as *BC* is to *EF*, that is, according to our notation, x is to $x-h$ as g to l; whence by multiplying extremes and means we have $gx-gh=lx$, and x, or the altitude of the cone, equal to $\frac{gh}{g-l}$. In like manner if from the value of x we subtract h, the altitude of the frustum, we shall find *Ad*, or the height of the cone to be cut off, equal to $\frac{lh}{g-l}$. Now the solid content of every cone is found by multiplying the base into a third part of it's altitude; therefore since the base of the cone *ABC* is $\frac{11gg}{14}$, and it's altitude $\frac{gh}{g-l}$, it's solid content will be $\frac{g^3}{g-l}\times\frac{h}{3}\times\frac{11}{14}$: in like manner the solid content of the cone *AEF* will be $\frac{l^3}{g-l}\times\frac{h}{3}\times\frac{11}{14}$: subtract the latter cone from the former, and there will remain the solid content of the frustum *BEFC* equal to $\frac{g^3-l^3}{g-l}\times\frac{h}{3}\times\frac{11}{14}$: but the fraction $\frac{g^3-l^3}{g-l}$ may be reduced to an integer by dividing the numerator by the denominator, and the quotient will be $gg+gl+ll$; therefore the solid content of the frustum *BEFC* will now be expressed thus, $\overline{gg+gl+ll}\times\frac{h}{3}\times\frac{11}{14}$. Whence we have the following canon:

Add the squares and the rectangle of the two given diameters together; multiply the sum into a third part of the given altitude, and the product will be the frustum of a pyramid of the same height having square bases whose sides are equal to the two diameters given; and as 14 *is to* 11 *so will this frustum be to the frustum sought.* Q. E. I.

N. B.

N. B. 1st. Since a cone differs nothing from a frustum of a cone whose lesser base is equal to nothing, if l be made equal to nothing in the foregoing canon, it ought to give the solid content of a cone whose height is h, and the diameter of whose base is g: and so it will; for then $\overline{gg+gl+ll}\times\frac{h}{3}\times\frac{11}{14}$ becomes $\frac{11gg}{14}\times\frac{h}{3}$.

2dly. Since a cylinder may be considered as a frustum of a cone whose bases are equal, if l be made equal to g in the foregoing canon, it ought to give the solid content of a cylinder whose height is h, and the diameter of whose base is g: and so we find it will; for $\overline{gg+gl+ll}\times\frac{h}{3}\times\frac{11}{14}$ in this case becomes $3gg\times\frac{h}{3}\times\frac{11}{14}=\frac{11}{14}ggh$.

3dly. If the lesser base of the frustum be supposed to be increased till it becomes equal to the greater; and if, on the other hand, the greater base be supposed to be diminished till it becomes equal to that which was the lesser base before, the solid content of the frustum will be the same as at the first; and therefore if the foregoing canon be just, it ought not to be altered by changing the quantities g and l one for the other: which is true; for $gg+gl+ll$ by this means only becomes $ll+lg+gg$, which is the same quantity.

In solving this last problem it is taken for granted that every cone is the third part of a cylinder having the same base and height; which may safely be done, since *Euclid* has demonstrated it in the 10th of the twelfth book of the Elements: but because *Euclid*'s doctrine of solids is not so easy to the imaginations of young beginners, I shall (in another place) give another demonstration of this proposition, independently of any thing that has here been said.

A LEMMA. *(Fig. 57.)*

355. *Let* ABC *be any curvilinear space comprehended between two streight lines* AB *and* AC *at right angles to each other, and a curve as* BC *concave towards* AB; *and from any two points* D *and* E *in the line* AB *let the lines* DF *and* EG *be drawn parallel to the base* AC, *and terminating in the curve at the points* F *and* G, *and compleat the parallelogram* DEGH: *then it is plain that the curvilinear space* DEGF *will be greater than the parallelogram,* DEGH. *But what I here propose to demonstrate is, that if the line* EG *be supposed to move towards* DF *in a position always parallel to itself, and at last to coincide with* DF, *the nearer* EG *approaches to* DF, *the nearer will the ratio of the curvilinear space* DEGF *to the parallelogram* DEGH *approach towards a ratio of equality, and that it will at last terminate in a ratio of equality when* EG *coincides with* DF.

For

For compleating the parallelogram *GHFK*, the parallelogram *DEKF* will be to the parallelogram *DEGH* upon the ſame baſe, as *DF* is to *EG*; therefore the curvilinear ſpace *DEGF* will be to the parallelogram *DEGH* in a leſs ratio than that of *DF* to *EG*; that is, though that ſpace be greater than this parallelogram, yet the ratio of the former to the latter is a leſs ratio than that of *DF* to *EG*: but the nearer the line *EG* approaches toward *DF*, the nearer will the ratio of *DF* to *EG* approach toward a ratio of equality, and it will at laſt become a ratio of equality when *EG* coincides with *DF*; therefore *a fortiori*, the ultimate ratio of the curvilinear ſpace *DEGF* to the parallelogram *DEGH* will be a ratio of equality.

Hence it follows, that if we ſuppoſe the line AB *to be divided into a certain number of parts, ſuch as* DE, *and theſe parts to be made the baſes of ſo many inſcribed parallelograms, ſuch as is the parallelogram* DG, *the more there are of theſe parallelograms, the nearer will the ſum of all the curvilinear ſpaces* DEGF, *that is the whole curvilinear ſpace* ABC, *approach towards the ſum of all the inſcribed parallelograms; and if we ſuppoſe the baſes of theſe parallelograms to be diminiſhed, and ſo their number to be augmented* ad infinitum, *they will make up the whole curvilinear ſpace* ABC; *ſo that the ſpace* ABC *will be to the ſum of all the inſcribed parallelograms ultimately in a ratio of equality.* For ſetting aſide the parts infinitely near the point of interſection *B*, which will be of no conſequence in the account, let the parallelogram *DEGH* be that which, of all the reſt, differs moſt from it's correſpondent curvilinear ſpace *DEGF*; and the conſequence will be that the curvilinear ſpace *ABC* will be to the ſum of all the inſcribed parallelograms in a leſs ratio than that of the ſpace *DEGF* to the ſpace *DEGH*: but even this ratio becomes at laſt a ratio of equality, when *DE* is infinitely ſmall, by this lemma: whence it follows *a fortiori*, that the ultimate ratio of the curvilinear ſpace *ABC* to the ſum of all the inſcribed parallelograms will be a ratio of equality.

I thought myſelf obliged to demonſtrate this propoſition, becauſe I have known it objected, that though the difference between any particular parallelogram and it's correſpondent curvilinear ſpace be allowed to be infinitely ſmall when the common baſe is ſo, yet how do we know but that an infinite number of theſe differences may amount to a finite quantity? and if ſo, then the whole curvilinear ſpace cannot be ſaid to be to the ſum of all the inſcribed parallelograms in a ratio of equality. To this I anſwer, that it muſt be the buſineſs of Geometry to determine whether an infinite number of theſe infinitely ſmall differences amount to a finite quantity or not; and by the demonſtration here given it appears that they do not, but that the ſum of all theſe differences actually diminiſhes as their number increaſes, and at laſt comes to nothing when their number is infinite.

A LEMMA.

A LEMMA. *(Fig. 57.)*

356. *Supposing the line* AB *still to keep it's place, let us imagine the whole space* ABC *to turn round it, so as to describe or generate a solid whose* axis *is* AB, *and the semidiameter of whose base is* AC, *and the inscribed parallelograms will at the same time by their common motion describe so many thin cylindric* laminæ, *which taken all together, will be less than the solid generated by the space* ABC; *but the more they are in number, the nearer they will approach to it, and their circular edges will at last terminate in the curve surface of the solid when their number is infinite.*

For compleating the parallelogram $GHFK$ as before, the *lamina* generated by the parallelogram DK will be to the *lamina* generated by the parallelogram DG as the square of DF is to the square of EG, all circles being as the squares of their semidiameters; therefore the *lamina* generated by the curvilinear space $DEGF$ will be to the *lamina* generated by the parallelogram DG in a less ratio than that of DF^2 to EG^2: but when D and E coincide, DF will be equal to EG, and the square of DF to the square of EG; therefore in this case, every particular cylindric *lamina* will be the same with a correspondent *lamina* of the solid; and *componendo*, all the cylindric *laminæ* will constitute the solid itself. This may also be further evident by applying the demonstration in the last lemma to this case. Therefore we need not scruple to suppose round solids, generated after the same manner as this is, to be made up of an infinite number of infinitely thin cylindric *laminæ*: *Nay even the cone itself may be considered as being so constituted: for if we suppose* BC *to be a streight line instead of a curve, the reasoning of the last article and this will equally succeed; in which case, the space* ABC *will be a triangle, and the figure generated a cone.*

If a solid be made up of a finite number of prismatic or cylindric laminæ, *decreasing in their diameters as they are further and further distant from the base, the surface of such a solid must necessarily ascend by steps: but the thinner these elementary* laminæ *are, (supposing their thinness to be compensated by a greater number,) the narrower and the shallower these steps will be, so as to terminate at last in a regular surface when their number is infinite.*

A THEOREM.

357. ***All isosceles cones of equal heights are as their bases; that is, the solid content of any one isosceles cone is to the solid content of any other of an equal height, as the base of the former cone is to the base of the latter.***

Note,

Note, that by an isosceles cone I mean a cone whose base is a circle, and whose vertex is every where equally distant from the circumference of the base; and by an isosceles pyramid is meant a pyramid having a regular polygon for it's base, and whose vertex is equally distant from all the angles of that polygon: lastly by isosceles prisms and cylinders are meant such as have equal and regular polygons and circles for their bases, and are so constituted, that a right line joining the centers of their two bases is perpendicular to them.

Let ABC (*Fig.* 58) be a right-angled triangle at B, and producing the base BC out to D, join AD; let also the line EFG be drawn any where within the triangle parallel to the base BCD, so as to cut AB in E, AC in F, and AD in G: then will EF be to BC as EG is to BD, since both are as AE to AB by similar triangles; therefore if they be taken alternately, EF will be to EG as BC to BD, and EF^2 to EG^2 as BC^2 to BD^2. This being allowed, let the triangle ABD be supposed to turn round the fixed side AB, so as to generate a cone whose *axis* is AB; then will the triangle ABC generate another cone having the same common altitude AB. Let both these cones be considered as constituted of an infinite number of infinitely thin cylindric *laminæ*, and let EF represent indifferently the semidiameter of any one of these *laminæ* belonging to the cone ABC; then will EG be the semidiameter of a correspondent *lamina* belonging to the cone ABD; and the *lamina* whose semidiameter is EF will be to the *lamina* whose semidiameter is EG, having the same thickness, as EF^2 is to EG^2, or (according to what is already demonstrated) as BC^2 is to BD^2; that is, every particular *lamina* of the cone ABC will be to a like *lamina* of the cone ABD as the base of the former cone is to the base of the latter; therefore *componendo*, the whole cone ABC will be to the whole cone ABD as the base of the former is to the base of the latter: but the planes ABC and ABD may be so constituted as to generate any two isosceles cones whatever that have equal heights; therefore if the heights of two isosceles cones be equal, these cones will be to each other as their bases. *Q. E. D.*

A Theorem.

358. *Every isosceles pyramid is equal to an isosceles cone of an equal base and height.*

Let P represent any isosceles pyramid, and C an isosceles cone of an equal base and height: I say then that the pyramid P will be equal to the cone C.

To demonstrate this, imagine the pyramid P to have a cone, as c, inscribed in it, so as to touch every side of the pyramid in so many lines of contact, and imagine the circumscribing pyramid, and consequently the in-

 scribed

ſcribed cone, to be conſtituted of an infinite number of infinitely thin *laminæ*; and every *lamina* of the circumſcribing pyramid will be to a correſpondent *lamina* of the inſcribed cone as the baſe of the pyramid is to the baſe of the cone. For the plane of every *lamina* that conſtitutes the pyramid will be a polygon ſimilar to the baſe, and the plane of every correſpondent *lamina* that conſtitutes the inſcribed cone will be a circle inſcribed in ſuch a polygon: therefore *componendo*, all the *laminæ* conſtituting the pyramid *P* will be to all thoſe that conſtitute the cone *c*, that is, the whole pyramid *P* will be to the whole cone *c* as the baſe of *P* is to the baſe of *c*: but the cone *c* is to the cone *C* of an equal height, as the baſe of *c* is to the baſe of *C*. Since then *P* is to *c* as the baſe of *P* is to the baſe of *c*, and *c* is to *C* as the baſe of *c* is to the baſe of *C*, it follows *ex æquo* that *P* is to *C* as the baſe of *P* is to the baſe of *C*: but the baſe of *P* is equal to the baſe of *C* by the ſuppoſition; therefore the pyramid *P* is equal to the cone *C*, having an equal baſe and altitude *Q. E. D.*

COROLLARY

Hence it follows, that whether cones be compared with cones, or cones with pyramids, or pyramids with pyramids, all ſuch as have equal heights will be to one another as their baſes. For cones are ſo by the laſt article, and pyramids are equal to cones having equal baſes and heights by this: I mean iſoſceles pyramids and iſoſceles cones.

SCHOLIUM.

As the ſolid content of a priſm or cylinder of a given perpendicular altitude depends upon the quantity, and not upon the figure of the baſe, ſo by the demonſtration of this propoſition it appears, that the ſolid content of an iſoſceles pyramid or cone of a given perpendicular altitude depends upon the quantity, and not upon the figure of the baſe: let the perpendicular altitude and the area of the baſe be the ſame, and the quantity of the ſolid will continue the ſame, whether that baſe be in the form of a triangle, or a ſquare, or a polygon, or a circle. Other pyramids and cones will be conſidered in another place.

A LEMMA.

359. *If from the center of any cube ſtreight lines be imagined to be drawn to all it's angles, the cube will by this means be diſtinguiſhed into as many equal iſoſceles pyramids as it has ſides, to wit 6, whoſe baſes will be in the ſides of the cube, and whoſe common vertex will be in the center.*

For if from a point above any right-lined plain figure, lines be drawn to all it's angles, and then the interſtices of theſe lines be imagined to be filled

filled up with triangular planes, the solid figure thus inclosed will be a pyramid; therefore by the lines above described, the cube will be distinguished into as many pyramids as it hath sides. And that these pyramids will be all equal and isosceles, is evident: for first, their bases will be all equal from the nature of the cube; and in the next place, their heights will be all equal from the nature of the center, which is supposed to be equally distant from all the sides of the cube; and lastly, as this center must also be equally distant from all it's angles, it follows that these pyramids must be all isosceles pyramids. *Q. E. D.*

COROLLARY.

Hence every one of these pyramids will be the sixth part of the whole cube

A THEOREM.

360. *Every isosceles pyramid or cone is a third part of an isosceles prism or cylinder having an equal base, and an equal perpendicular height.*

Let a be the perpendicular altitude of any isosceles pyramid or cone, and let b be the area of it's base, both taken according to the directions given in art. 353: imagine also a cube whose side, that is the side of whose square base, is $2a$; then will $4a^2$ be the area of the base, and $8a^3$ the solid content of this cube: and if from the center of the cube lines be imagined to be drawn to the four angles of the base, they will be the outlines of an isosceles pyramid whose base is the same with the base of the cube, to wit, $4a^2$, and whose perpendicular altitude is a; whence the solid content of this pyramid will be $\frac{8a^3}{6}$ or $\frac{4a^3}{3}$, by the corollary in the last article: but as this pyramid has the same height a with the pyramid proposed, these two pyramids will be to one another as their bases, by the corollary in the last article but one: hence the solid content of the pyramid proposed will easily be had by saying, as $4a^2$ the base of the pyramid within the cube, is to b the base of the pyramid proposed, so is $\frac{4a^3}{3}$ the solid content of the former, to a fourth, to wit $\frac{ab}{3}$, which will be the solid content of the latter; therefore the solid content of an isosceles pyramid or cone whose base is b, and whose altitude is a, is found to be $\frac{ab}{3}$: but the solid content of an isosceles prism or cylinder having an equal base and height is ab; therefore every isosceles pyramid or cone is a third part of an isosceles prism or cylinder having an equal base and an equal perpendicular altitude. *Q. E. D.*

COROLLARY 1.

Hence the solid content of an isosceles pyramid or cone is found by multiplying the area of the base into a third part of the perpendicular altitude; or else by multiplying the area of the base into the whole altitude, and then taking a third part of the product.

COROLLARY 2.

Hence also it follows that all isosceles pyramids and cones upon equal bases are to one another as their heights.

A LEMMA.

361. *If a pyramid of any kind be cut by a plane parallel to it's base, the quantity of the section, or (which is all one) the quantity of the base of the pyramid cut off, will always be the same, let the figure of the pyramid be what it will, so long as the base and perpendicular altitude of the whole pyramid, and the perpendicular altitude of the pyramid cut off continue the same: in which case, the perpendicular distance of the plane of the section from the plane of the base will also be the same.* (See *Fig.* 59.)

Let A be the vertex of the pyramid, and let BC be any one side of the base; let the lines AB and AC be cut by the plane of the section in the points D and E respectively, and let AFG be the perpendicular altitude of the whole pyramid, cutting the plane of the section in F, and the plane of the base in G, both produced if need be: join FD, FE, GB, GC: then since the base of the pyramid cut off will always be similar to the base of the whole pyramid, whereof DE and BC are correspondent sides; and since all similar plain figures are to each other as the squares of their correspondent sides by the 20th of the sixth book of the Elements, it follows that the base whose side is DE will be to the base whose side is BC as DE^2 to BC^2, that is, by similar triangles, as AD^2 is to AB^2, or as AF^2 is to AG^2. Since then as AG^2 is to AF^2 so is the base of the whole pyramid to the base of the pyramid cut off, so long as the three first continue the same, the last must also continue the same. *Q. E. D.*

COROLLARY.

Since the number of sides of the pyramid is not concerned in the demonstration of this proposition, which will be equally true, be the number of sides what it will, it must also be true of the cone, which is nothing else but a pyramid of an infinite number of sides, let the shape of the cone be what it will; that is, whether AG *the perpendicular altitude of the cone passes through the center of the base or not.*

A THEO-

A THEOREM.

362. *If a prism or cylinder of any kind be described by the motion of a plain figure ascending uniformly in a horizontal position to any given height, the quantity of the solid thus generated will be the same, whether the describing plane ascends directly or obliquely to the same height; and consequently all prisms and cylinders of what kind soever, that have equal bases and equal perpendicular heights, are equal, whether they stand upon those bases erect or reclining.*

For the better conceiving of this, let the describing plane be made, not to ascend all the way, but sometimes to ascend perpendicularly, and sometimes to move laterally or edgeway, and that by turns: then it is plain that the quantity of solid space, or rather the sum of all the solid spaces thus described, will amount to no more than if the describing plane had ascended all the way perpendicularly to the same height. Let the times of these alternate motions wherein they are performed be diminished and their number be increased *ad infinitum*, and they will terminate at last in an uniform oblique motion, and the solid generated by this motion will be equal to a solid generated by a perpendicular motion of the same plane to the same height. *Q. E. D.*

N. B. What has here been demonstrated concerning prisms and cylinders, may be further illustrated by sliding a pack of cards, or a pile of halfpence out of an erect into an oblique posture; whereby it may easily be seen, that neither the base, nor the perpendicular altitude, nor the quantity of the solid can be affected by this change of posture; but the finer, that is the thinner these constituent *laminæ* are, the nearer they will represent an oblique solid.

A THEOREM.

363. *All pyramids and cones of what kind soever that have equal bases and equal perpendicular heights are equal.*

To evince this, let us imagine two plain figures (whether similar or dissimilar to each other it matters not) to rise together from the same level, one directly, and the other obliquely, but both in a horizontal position, and always upon the same level; and let these planes be imagined not to retain all along their first magnitude (as was supposed in the last article) but to lessen by degrees as they rise, so as by their motion to describe tapering figures, and at last to vanish each in a point: then it is easy to see, that if the tapering figures thus described be pyramids or cones having equal bases and equal perpendicular heights, these describing planes must not only be equal to each other at first, and vanish at equal heights,

but

but they must lessen so together as to be equal to each other at all other equal altitudes whatever: this is evident from the last article but one: and therefore the solids described by them must necessarily be equal. *Q. E. D.*

COROLLARY.

Hence it follows, that whatever we have demonstrated concerning isosceles pyramids, cones, prisms and cylinders with respect to their proportion one to another, will be equally true of all others, whatever shape or posture they may be in: as, that all pyramids and cones of the same height are to each other as their bases, that all pyramids and cones upon equal bases are as their heights, and that every pyramid or cone is a third part of a prism or cylinder having an equal base, and an equal perpendicular altitude.

A LEMMA. (*Fig.* 60.)

364. *Let* ABCD *be a square whose base is* AD, *and whose diagonal is* AC; *and upon the center* A, *and with the* radius AB, *describe the quadrant or quarter of a circle* BAD: *draw also the line* EFGH *or* EGFH *any where within the square, parallel to the base* AD, *cutting the side* AB *in* E, *the quadrant* BD *in* F, *the diagonal* AC *in* G, *and the opposite side* CD *in* H, *and join* AF: *I say then that the square of* EF *and the square of* EG *put together will always be equal to the square of* EH.

For the triangles ABC and AEG are similar, as having one angle at A in common, and the angles at B and E right; therefore EG will be to EA as BC is to BA; but BC is equal to BA, from the nature of a square; therefore EG will be equal to EA, and EG^2 to EA^2, and EF^2+EG^2 to $EF^2+EA^2=AF^2=AD^2=EH^2$, that is, $EF^2+EG^2=EH^2$. *Q. E. D.*

A THEOREM.

365. *Every sphere is two thirds of a circumscribing cylinder, that is, a cylinder that will just contain it.*

For supposing all things as in the last article, (see *Fig.* 60,) let the square $ABCD$ be now supposed to turn round it's fixed side AB, so that the square may generate a cylinder, the quadrant a hemisphere, and the triangle ABC an inverted cone; and let this cylinder, and consequently the cone and hemisphere be considered as consisting of an infinite number of infinitely thin cylindric *laminæ*: then if EH represents the semidiameter of any one of these *laminæ* belonging to the cylinder, EG will be the semidiameter of so much of this *lamina* as lies within the cone, and EF the semidiameter of so much as lies within the hemisphere: and because (by the last article) the square of EF and the square of

E

COROLLARIES.

368. 1st. *As the diameter of a circle is to the circumference, that is, as 7 to 22 nearly, so is the square of the diameter of any sphere to it's surface.* For supposing the diameter of a circle to be to the circumference as 1 to c, and putting d for the diameter of any sphere, cd will be the circumference of a great circle of that sphere, since as 1 is to c so is d to cd; multiply then $\frac{cd}{2}$ the semicircumference, into $\frac{d}{2}$ the *radius*, and you will have $\frac{cdd}{4}$ the area of a great circle; therefore four great circles, or the surface of the sphere, will be cdd: but as 1 is to c so is dd to cdd; therefore &c.

2d. Whence it follows, that *The surface of every sphere is equal to the product of the circumference of a great circle multiplied into the diameter of the sphere.* For retaining the notation of the last article, cdd the surface of the sphere is equal to cd the circumference of a great circle multiplied into d the diameter.

3d. *The surface of every sphere is equal to the convex surface of a circumscribed cylinder.* For if a concave cylinder without it's two bases be slit, and then opened into a plane, the figure of that plane will be a parallelogram, whose base will be that line which before was the circumference of the base of the cylinder, and whose height will be the same with that of the cylinder; therefore as the area of a parallelogram is found by multiplying the base into the height, the surface of every cylinder must be found by multiplying the circumference of the base into the height of the cylinder: but the circumference of a cylinder circumscribed about a sphere is equal to the circumference of a great circle of the sphere, and the height of such a cylinder is equal to the diameter of the sphere; therefore the convex surface of the cylinder will be equal to the circumference of a great circle of the sphere multiplied into the diameter, which by the last corollary is the surface of the inscribed sphere.

4th. *The solid content of every sphere is equal to the product of it's surface multiplied into a third part of the* radius, *or the* radius *into a third part of the surface.* This is evident from art. 366.

5th. *As six times the diameter of a circle is to the circumference, that is, as 42 is to 22 or 21 to 11 nearly, so is the cube of the diameter of any sphere to it's solid content.* For if we suppose the diameter of a circle to be to the circumference as 1 to c, the surface of a sphere whose diameter is d will be cdd by the first corollary; and this surface multiplied into a third part of the *radius*, or into a third part of $\frac{d}{2}$, which is $\frac{d}{6}$,

gives $\frac{cd^3}{6}$ the ſolid content of the ſphere: but as 6 is to c ſo is d^3 to $\frac{cd^3}{6}$; therefore as ſix times the diameter of a circle is to the circumference ſo is the cube of the diameter of any ſphere to it's ſolid content.

6th. *The ſurfaces of all ſpheres are as the ſquares, and the ſolid contents as the cubes of their diameters or ſemidiameters.* For ſuppoſing the diameter of any circle to be to the circumference as 1 to c, and ſuppoſing d and e to be the diameters of two ſpheres, the ſurfaces will be cd^2 and ce^2 by the firſt corollary, and the ſolid contents will be $\frac{cd^3}{6}$ and $\frac{ce^3}{6}$ by the laſt: but cd^2 is to ce^2 as d^2 is to e^2, or as $\frac{d^2}{4}$ is to $\frac{e^2}{4}$; and $\frac{cd^3}{6}$ is to $\frac{ce^3}{6}$ as d^3 is to e^3, or as $\frac{d^3}{8}$ is to $\frac{e^3}{8}$.

To ſhew the uſe of the properties of the ſphere above deſcribed, I ſhall add the following problems.

PROBLEM I.

369. *To find how many acres the ſurface of the whole earth contains.*

Let the diameter of a circle be to the circumference as d to c, and let e be the circumference of the earth; then will $\frac{de}{c}$ be it's diameter, and $\frac{de^2}{c}$ it's ſurface by the ſecond corollary in the laſt article. Now the circumference of the earth is 131630573 Engliſh feet, or 24930 Engliſh miles nearly, allowing 5280 feet to a mile: therefore if we make $e = 24930$, we ſhall have $e^2 = 621504900$. Now the numbers 7 and 22 are ſcarce exact enough to expreſs the proportion of the diameter of a circle to the circumference in company with ſo large a number as e^2; let us therefore uſe that of 113 to 355, which we have elſewhere ſhewn (ſchol. 1 in art. 179) to be much more exact; that is, let $d = 113$ and $c = 355$, and $\frac{de^2}{c}$ or the ſurface of the earth will be 197831137 ſquare miles: but every ſquare mile contains 640 acres; therefore if the foregoing number of ſquare miles be multiplied by 640, the product 126611927680 will be the number of acres required.

N. B. To be more exact in any computation than the *data* on which it is founded, can be to little or no purpoſe.

PROBLEM 2.

370. *What must be the diameter of a concave sphere that will just hold an English gallon?*

By the fifth corollary in art. 368, as 11 is to 21 so is the solid content of any sphere to the cube of it's diameter: but the solid content of our sphere is 282 cubic inches or an English gallon by the supposition; therefore the cube of it's diameter will be $538\frac{4}{11}$, the cube root whereof 8.135 will be the diameter itself.

N. B. The extraction of the cube root is taught in most books of Arithmetic, and depends on the nature of a binomial, as doth the extraction of the square root; and therefore whoever sees the reason of the latter, may (without much difficulty) reason himself into the former: but the extraction of the roots of all simple powers will best be performed by the help of logarithms, as will be shewn hereafter when we come to treat of the nature and properties of those numbers.

PROBLEM 3.

371. *To find the weight of a globe of water of an inch diameter when weighed in air, upon a supposition that a cubic foot of common rain water when weighed in air at a middle height of the barometer weighs just 76 pounds Troy.*

Note, that the pound Troy contains 12 ounces, every ounce 20 pennyweights, and every pennyweight 24 grains.

If a cubic foot, or $12\times12\times12$ cubic inches weigh 76 pounds, the weight of one cubic inch will be $\frac{76}{12\times12\times12}$ of a pound, or $\frac{76\times12\times20\times24}{12\times12\times12}$ grains, that is, (dividing both the numerator and denominator by $12\times12\times4$) $\frac{760}{3}$ grains. Let now the diameter of a circle be to the circumference as d to c; and by the fifth corollary in art. 368, as $6d$ is to c so is the cube of the diameter of any sphere to it's solid content: but the cube of the diameter of a sphere is the solid content of a cube whose side is equal to the diameter; therefore as $6d$ is to c so is the solid content of a cubic inch to the solid content of a globe of an inch diameter, and so is $\frac{760}{3}$ the weight of a cubic inch of water, to $\frac{760c}{18d}$ the weight of a globe of water of an inch diameter when weighed in air. Let $d=113$ and $c=355$, and we shall have $\frac{760c}{18d}=132.645$, or $132\frac{20}{31}$; there-

 fore

fore a globe of water of an inch diameter when weighed in air, weighs $132\frac{20}{31}$ grains.

In this computation it is all along taken for granted that *The weights of homogeneous bodies in air are in the ſame proportion one to another as* in vacuo, *and conſequently as their ſolid contents*; which is true, though in the caſe of heterogeneous bodies, or rather bodies of different ſpecific gravities it be otherwiſe. But to put this matter beyond all diſpute, let a and b be the weights of any two bodies A and B of the ſame kind (ſuppoſe of water) when weighed *in vacuo*, and let water be 860 times as heavy as air; then will $\frac{a}{860}$ be the weight of ſo much air as is equal in bulk to the body A, and $\frac{b}{860}$ the weight of ſo much air as is equal in bulk to the body B. Now it is demonſtrated in Hydroſtatics, that the weight of a body in air is not it's true weight, or weight it would have *in vacuo*, but the exceſs of it's true weight above the true weight of an equal bulk of air; therefore the weight of the body A when weighed in air, will be $a-\frac{a}{860}=\frac{859a}{860}$; and for the ſame reaſon the weight of B when weighed in air will be $\frac{859b}{860}$: but $\frac{859a}{860}$ is to $\frac{859b}{860}$ as a is to b; therefore the weights of homogeneous bodies in air will be in the ſame proportion one to another as their true weights *in vacuo*; but the weights of homogeneous bodies *in vacuo* are as their magnitudes or ſolid contents; and therefore their weights in air muſt be in the ſame proportion. *Q. E. D.*

PROBLEM 4.

372. *To find the diameter of a globe by weighing it firſt in air, and then in water, without any regard to it's weight* in vacuo.

Let d *be the diameter ſought in inches and parts of an inch; let* p *be the unknown weight of the globe* in vacuo, q *it's known weight in air, and* r *it's known weight in water, the weight* p *being ſuppoſed, and the weights* q *and* r *being taken in grains:* then from a principle in Hydroſtatics mentioned in the laſt article it is evident, that $p-q$ will be the weight *in vacuo* of a globe of air whoſe diameter is d, and that $p-r$ will be the weight *in vacuo* of a globe of water of the ſame diameter: ſubtract therefore the former weight from the latter, and the remainder $q-r$ will expreſs the weight of a globe of water whoſe diameter is d when weighed in air: but the weights of all homogeneous bodies in air are in the

the ſame proportion as their ſolid contents by the laſt article; and the ſolid contents of all ſpheres are as the cubes of their diameters by the ſixth corollary in art. 368: ſince then the weight of a globe of water of an inch diameter was found in the laſt article to be $132\frac{20}{31}$ or $\frac{4112}{31}$ grains, we have the following proportion; *As* $\frac{4112}{31}$ *the weight in air of a globe of water whoſe diameter is* 1 *inch, is to* q—r *the weight in air of a globe of water whoſe diameter is* d, *ſo is* 1 *the cube of the diameter of the former globe to* d^3 *the cube of the diameter of the latter: therefore* $d^3 = \overline{q-r} \times \frac{31}{4112}$, *and* d *the diameter ſought will be the cube root of the ſame quantity.* Q. E. I.

As for example; ſuppoſe a globe to weigh $156\frac{1}{4}$ grains in air, and 77 in water; then will q equal $156\frac{1}{4}$, $r=77$, $q-r=79.25$, and $\overline{q-r} \times \frac{31}{4112} = .5974587$, whoſe cube root .84224 will be the diameter of the globe in decimal parts of an inch. See experiment 1 in the ſcholium to the 40th propoſition of the ſecond book of *Newton*'s *Principia*.

The *praxis* in logarithms is very eaſy: for *If from a third part of the logarithm of* q—r *be ſubtracted the logarithm* 0.7075636, *the remainder will be the logarithm of the diameter ſought.*

The chief excellency of this ſtatical method is, that it is not only much more exact than any other that can be made uſe of for this purpoſe, but if the globe under conſideration be not a perfect ſphere, it gives a middle diameter, that is, ſuch a diameter as the ſame body would have if it was formed into a perfect ſphere: nay if the body propoſed be in form of a cube or of any other ſolid whatever, regular or irregular, this method finds the diameter of a globe of the ſame magnitude.

By a like way of thinking may alſo be diſcovered a proportion for determining the ſolid content of any body propoſed in cubic inches and parts of a cubic inch thus: let q *be the weight of any body in air, and* r *it's weight in water, both taken in grains: then as* $\frac{760}{3}$ *grains, the weight in air of a cubic inch of water, is to* q—r *the weight in air of a body of water equal in magnitude to the body propoſed, ſo is* 1 *the ſolid content of the former, to* $\overline{q-r} \times \frac{3}{760}$ *the ſolid content of the latter:* thus the ſolid content of the globe in the foregoing example is .31283 of a cubic inch.

If the globe whose diameter, or body whose solid content is required, be lighter than water, let r *be the force necessary to keep it under water; and then the weight of that body in water may be said to be* —r, *in which case the sign of* r *must be changed in the proportions above laid down, that is,* q+r *must be used instead of* q—r.

Of the Spheroid.

373. *If a sphere be resolved into an infinite number of infinitely thin cylindric* laminæ, *and then these* laminæ, *retaining their circular figure, be all increased or all diminished in the same proportion, they will constitute a figure called a spheroid; and it is said to be prolate or oblong, according as these constituent* laminæ *are increased or diminished.* This a learner, who is unacquainted with the nature of the *ellipsis*, may (if he pleases) take for the definition of a spheroid.

From the definition here given it follows

1st, that *Every spheroid is to a sphere upon the same* axis, *as any one* lamina *in the former is to a like* lamina *in the latter from whence it was derived; or as any number of* laminæ *in the former is to the same number of like* laminæ *in the latter, that is, as any portion of the former comprehended between two parallel planes perpendicular to it's* axis, *is to a like portion of the latter.*

2dly it follows, that *Every spheroid, as well as every sphere, is two thirds of a circumscribing cylinder.* For though a spheroid be greater or less than a sphere upon the same *axis*, the cylinder circumscribed about the spheroid will be proportionably greater or less than the cylinder circumscribed about the sphere: for having the same length, they will be as their bases; therefore the spheroid will have the same proportion to a cylinder circumscribed about it, as the sphere hath to a cylinder circumscribed about the sphere.

A LEMMA.

374. *The chord of any circular arc is a mean proportional between the versed sine of that arc and the diameter.*

Let *ABC* (*Fig.* 61) be a semicircle whose diameter is *AC*, and assuming any arc as *AB*, draw the streight line *AB*, which is it's chord; draw also *BD* perpendicular to the diameter *AC* in *D*, and the intercepted line *AD* is called the versed sine of the arc *AB*. What we are then to demonstrate is, that the chord *AB* is a mean proportional between the versed sine *AD* and the whole diameter *AC*: and this is easily done by drawing the other chord *BC*; for then the triangle *ABC* will be right-angled at *B*, as being in a semicircle, and consequently will be similar to the right-angled triangle *ADB*; whence *AD* will be to *AB* as *AB* to *AC*. *Q. E. D.*

PRO-

PROBLEM 5.

375. *To find the solid content of a frustum of a hemisphere or hemispheroid comprehended between a great circle perpendicular to it's* axis *and any other lesser circle parallel to it, having these two opposite bases and the height of the frustum given.*

N. B. As $\square AD$ is sometimes used for the square of AD, or a square whose side is AD, so in our notation in this and some of the following articles, we shall not scruple to use $\odot AD$ for the area of a circle whose semidiameter is AD, $2\odot AD$ for two such circles, &c.

Let $ABCD$ (*Fig.* 60) be a square whose base is AD and diagonal AC; and upon the center A and with the *radius* AB describe the quadrant BAD; draw also the line $EFGH$ any where within the square parallel to AD, cutting AB in E, the quadrant in F, the diagonal in G, and the opposite side CD in H. This done, imagine the whole figure to turn round it's fixed side AB: then will the square generate a cylinder, the quadrant a hemisphere, the triangle ABC an inverted cone, and the curvilinear space $AEFD$ such a frustum of an hemisphere as we are to find the solid content of, having given AD and EF the semidiameters of the two opposite bases, and AE the height of the frustum.

In the 365th article by the help of this construction it was demonstrated, that the hemisphere generated by the quadrant ABD and the cone generated by the triangle ABC were together equal to the cylinder generated by the square $ABCD$; and the reasons there given for such an equality, equally prove that the frustum generated by the space $AEFD$ and the cone generated by the triangle AEG will both together be equal to the cylinder generated by the parallelogram $AEHD$: but the cone generated by the triangle AEG is equal to $\odot EG \times \frac{AE}{3}$; and the cylinder generated by the parallelogram $AEHD$ is equal to $\odot AD \times AE$ $= 3\odot AD \times \frac{AE}{3} = \overline{2\odot AD + \odot EH} \times \frac{AE}{3}$: therefore if f be put for the solid content of the frustum, we shall have the following equation, $f + \odot EG \times \frac{AE}{3} = \overline{2\odot AD + \odot EH} \times \frac{AE}{3}$; transpose $\odot EG \times \frac{AE}{3}$, and then we shall have $f = \overline{2\odot AD + \odot EH - \odot EG} \times \frac{AE}{3}$: but by the 364th article, and the sixth corollary in the 347th, $\odot EH = \odot EF + \odot EG$; therefore $\odot EH - \odot EG = \odot EF$: substitute $\odot EF$ instead of $\odot EH - \odot EG$ in the foregoing equation $(f = \overline{2\odot AD + \odot EH - \odot EG}$ $\times$

$\times\frac{AE}{3}$) and you will have $f = \overline{2\odot AD + \odot EF} \times \frac{AE}{3}$: this is upon a supposition that the solid proposed is a frustum of a hemisphere. Let us now suppose the solid f to consist of an infinite number of infinitely thin cylindric *laminæ* parallel to it's base, and then that these *laminæ*, retaining their circular figure, be all diminished in some given proportion, suppose in the proportion of r to s; then it is plain that the solid f will degenerate into a frustum of an hemispheroid, and that it will be diminished in the proportion of r to s; but then the quantity $\overline{2\odot AD + \odot EF} \times \frac{AE}{3}$ will also be diminished in the same proportion; and therefore f will still be equal to $\overline{2\odot AD + \odot EF} \times \frac{AE}{3}$; whence we have the following theorem for finding the solid content of the frustum proposed, whether it be a frustum of a hemisphere or hemispheroid.

To twice the area of the greater base add the area of the less; multiply the sum by a third part of the altitude of the frustum, and the product will be it's solid content. Q. E. I.

PROBLEM 6.

376. *To find the convex surface of any segment of a sphere whose base and height are given.* (Fig. 60.)

Retaining the construction of the last article, and supposing what was there proved, if from the hemisphere generated by the space ABD be subtracted the frustum generated by the space $AEFD$, there will remain a segment of the sphere generated by the space BEF; and if to this segment again be added the cone generated by the triangle AEF, they will both together constitute a sector of the sphere generated by the space ABF; and lastly if the solid content of this spherical sector be applied to or divided by a third part of the *radius* AD, the plane or quotient thence arising will be equal to the convex surface generated by the arc BF, which is here proposed to be determined. For as every sphere is equal to a cone whose base is it's surface and whose altitude is it's *radius*, (see art. 366,) so (and for the same reason) must every sector of a sphere be equal to a cone whose base is the spherical part of it's surface, and whose altitude is the *radius*. Now the hemisphere generated by the space ABD being two thirds of a cylinder of the same base and height, as was demonstrated in art. 365, it's solid content will be expressed by $2\odot AD \times \frac{AB}{3} = 2\odot AD \times \frac{AE}{3} + 2\odot AD \times \frac{EB}{3}$; and the solid content of

of the fruſtum generated by the ſpace $AEFD$ was $2\odot AD\times\frac{AE}{3}+\odot EF\times\frac{AE}{3}$; ſubtract the latter from the former, and there will remain the ſegment generated by the ſpace BEF equal to $2\odot AD\times\frac{EB}{3}-\odot EF\times\frac{AE}{3}$; add to this the cone generated by the triangle AEF, whoſe content is $\odot EF\times\frac{AE}{3}$, and you will have the ſpheric ſector generated by the ſpace ABF equal to $2\odot AD\times\frac{EB}{3}$. Let the diameter of a circle be to the circumference as 1 to c, and $2AD\times c$ will be the circumference of a great circle, whoſe half $AD\times c$ multiplied into AD the *radius*, will give $AD^2\times c$ for the area of a great circle; therefore $\odot AD=AD^2\times c$, and $2\odot AD\times\frac{EB}{3}$, or the content of the ſector, will be $2AD^2\times c\times\frac{EB}{3}$; but EB is the verſed ſine of the arc BF; and therefore if we put l for the chord of that arc, we ſhall have $2AD\times EB=l^2$ by the laſt article but one; and the ſolid content of the ſector will now be $l^2\times c\times\frac{AD}{3}$; divide by $\frac{AD}{3}$, and you will have the ſurface generated by the arc BF equal to $l^2\times c$: but as $AD^2\times c$ was equal to $\odot AD$, ſo will $l^2\times c$ be equal to $\odot l$, that is, to a circle whoſe *radius* is the chord of the arc BF: therefore *The ſurface of every ſegment of a ſphere is equal to a circle whoſe* radius *is the diſtance of the pole, or vertical point of the ſegment, from the circumference of it's baſe.*

What has here been determined concerning the convex ſurface of a ſegment of a ſphere agrees intirely with what was determined in art. 367 concerning the ſurface of a whole ſphere. For if we ſuppoſe the arc BF to be a ſemicircle, it's chord will then be a diameter, and the ſurface generated by this arc will be the ſurface of the whole ſphere; and therefore the ſurface of this ſphere will be equal to a circle whoſe *radius* is the diameter of the ſphere, that is, $2AD$: but a circle whoſe *radius* is $2AD$ is quadruple of a circle whoſe *radius* is AD, becauſe all circles are as the ſquares of their ſemidiameters; therefore the ſurface of every ſphere is equal to four great circles of the ſame, as was there demonſtrated.

PROBLEM 7.

377. *To find the solid content of a cask, when it is a portion of a sphere or spheroid, terminated at each end by two equal and parallel circles at right angles with it's* axis, *having given the diameter at the head, the diameter at the bung, and the length of the vessel.* (Fig. 60.)

Retaining still the construction of the 375th article, if two such frustums as were there generated by the revolution of the space $AEFD$ about the *axis* AE be put together, by applying the greater base of one to the greater base of the other, they will form such a solid as is here described: but in the article above-quoted, the content of the frustum generated by the space $AEFD$ was found to be $\overline{2\odot AD+\odot EF}\times\frac{AE}{3}$: therefore the content of the solid proposed will be $\overline{2\odot AD+\odot EF}\times\frac{2AE}{3}$, which $\frac{2AE}{3}$ will be one third part of the whole length of the solid. Call the diameter of the solid where it is greatest, g, and where it is least, l; that is, make $g=2AD$, and $l=2EF$, and you will have $\odot AD=g^2\times\frac{11}{14}$, and $\odot EF=l^2\times\frac{11}{14}$ by the second corollary in art. 347; and the content of our solid according to this notation will be $\overline{2g^2+l^2}\times\frac{2AE}{3}\times\frac{11}{14}$. Let now the solid degenerate from a portion of a sphere to a like portion of a spheroid, and the quantity $\overline{2g^2+l^2}\times\frac{2AE}{3}\times\frac{11}{14}$ will be diminished in the same proportion with the solid itself, and therefore will still be equal to it: whence we have the following theorem for measuring the cask proposed.

To twice the square of the diameter at the bung add the square of the diameter at the head; multiply the sum by a third part of the length of the cask, and then say as 14 *is to* 11 *so is this last product to the content required.*

N. B. If the two circles whereby the vessel is terminated at each end be unequal, the two frustums must be measured apart, and then added together.

If the content of the vessel be required in ale gallons, the diameters g and l must be taken in inches; and instead of using the proportion of 14 to 11, the quantity $\overline{2g^2+l^2}\times\frac{2AE}{3}$ must be divided by 359, and the quotient will be the number of gallons the vessel will contain. See art. 353.

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ELEMENTS of ALGEBRA

BOOK IX. IN FOUR PARTS.

I. Of Powers and their Indexes; and of *Newton*'s method of evolving a Binomial.

II. Of Logarithms, their uſe, and the beſt methods of computing them.

III. Of *Newton*'s invention of Diviſors.

IV. Of the Arithmetic of Surd Quantities.

PART I.

Of powers and their indexes.

378. THE indexes of powers have been already conſidered, ſo far as they ſerve for a ſort of ſhort hand writing in Algebra; but the incomparable *Newton* has very much enlarged our views with reſpect to theſe indexes or exponents, inſomuch that it is by their means chiefly, that ſo many excellent, uſeful and comprehenſive theorems have been diſcovered both in Algebra and Geometry, and more particularly in the doctrine of Fluxions. This ſort of notation therefore I ſhall now endeavour further to explain, in my obſervations upon the following ſmall table.

Powers without their indexes.

$xxxxx.\quad xxxx.\quad xxx.\quad xx.\quad x.\quad 1.\quad \frac{1}{x}.\quad \frac{1}{xx}.\quad \frac{1}{xxx}.\quad \frac{1}{xxxx}.\quad \frac{1}{xxxxx}.$

Powers with their indexes.

$x^5.\quad x^4.\quad x^3.\quad x^2.\quad x^1.\quad x^0.\quad x^{-1}.\quad x^{-2}.\quad x^{-3}.\quad x^{-4}.\quad x^{-5}.$

This

This table consists of two rows, whereof the upper is a series of powers expressed without their indexes, the common root or fundamental quantity being x; the lower expresses the same powers by the help of their indexes.

OBSERVATIONS.

379. 1st. *By this table it appears that every subsequent power is the quotient of the next before it divided by the common root* x, *and that every subsequent index is generated by subtracting unity from the next before it.* Thus x^2 divided by x gives x, x divided by x gives 1, 1 divided by x gives $\frac{1}{x}$, $\frac{1}{x}$ divided by x gives $\frac{1}{xx}$ &c: thus again, $2-1=1$, $1-1=0$, $0-1=-1$, $-1-1=-2$ &c. Since then each row exhibits a regular series, it follows that the negative indexes have the same right to express the powers they belong to as the affirmative ones, and that x^{-2} represents $\frac{1}{xx}$ upon the same foundation that x^2 represents xx.

2dly. *Therefore whatever number is the index of any power, it's negative will be the index of the reciprocal of that power, or of unity divided by that power.* Thus if 2 be the index of xx, -2 will be the index of $\frac{1}{xx}$; if 1 be the index of x, -1 will be the index of $\frac{1}{x}$; and so of the rest.

3dly. *In all cases whatever, the addition of indexes answers to the multiplication of the powers to which they belong; that is, if any two powers of the same quantity be multiplied together, the index of the multiplicator added to the index of the multiplicand will give the index of the product.* Thus x^2 multiplied into x^3 gives x^5, as $xx\times xxx$ gives $xxxxx$: thus $x^2\times x^{-3}$ gives x^{-1}, as $xx\times\frac{1}{xxx}$ gives $\frac{1}{x}$: thus $x^{-2}\times x^{-3}$ gives x^{-5}, as $\frac{1}{xx}\times\frac{1}{xxx}$ gives $\frac{1}{xxxxx}$: thus $x^2\times x^{-2}$ gives x^0, as $xx\times\frac{1}{xx}$ gives 1: thus $x^3\times x^0$ gives x^3, as $xxx\times 1$ gives xxx.

4thly. *In like manner the subtraction of indexes answers to the division of powers; that is, if any power of any quantity be divided by a power of the same quantity, the index of the divisor subtracted from the index of the dividend leaves the index of the quotient.* Thus x^3 divided by x^2 quotes x^1, as xxx divided by xx quotes x: thus x^2 divided by x^{-3} quotes x^5, as xx divided by $\frac{1}{xxx}$ quotes $xxxxx$: thus x^{-2} divided by x^{+3} quotes x^{-5},

as $\frac{1}{xx}$ divided by xxx quotes $\frac{1}{xxxxx}$: thus x^{-2} divided by x^{-3} gives x^{1}, as $\frac{1}{xx}$ divided by $\frac{1}{xxx}$ gives x: thus x^{0} divided by x^{-2} gives x^{2}, as 1 divided by $\frac{1}{xx}$ gives xx: laſtly, x^{2} divided by x^{2} gives x^{0}, as xx divided by xx gives 1.

5thly. *If the index of any power be multiplied by 2, 3, 4 &c, the product will be the index of the ſquare, cube, ſquare-ſquare &c of that power: and therefore if the index of any power be divided by 2, 3, 4 &c, the quotient will be the index of the ſquare root, cube root, ſquare-ſquare root &c of that power.* Thus the ſquare of x^{2} is x^{4}, it's cube x^{6}, it's ſquare-ſquare x^{8}: thus again, the ſquare root of x^{12} is x^{6}, it's cube root x^{4}, it's ſquare-ſquare root x^{3} &c: thus the ſquare root of x or x^{1} is $x^{\frac{1}{2}}$, it's cube root $x^{\frac{1}{3}}$, it's ſquare-ſquare root $x^{\frac{1}{4}}$ &c: thus the ſquare root of $\frac{1}{x}$ or x^{-1} is $x^{-\frac{1}{2}}$, it's cube root $x^{-\frac{1}{3}}$, it's ſquare-ſquare root $x^{-\frac{1}{4}}$ &c: thus $x^{\frac{2}{3}}$ ſignifies the cube root of x^{2}, $x^{\frac{3}{4}}$ the ſquare-ſquare root of x^{3}. And univerſally, $x^{\frac{m}{n}}$ ſignifies that root of x^{m} whoſe index is n; as if $y^{n}=x^{m}$, then y is ſaid to be that root of x^{m} whoſe index is n, and muſt be expreſſed by $x^{\frac{m}{n}}$; and therefore if in any caſe $x^{m}=y^{n}$, it will be a good inference to ſay that y is equal to $x^{\frac{m}{n}}$, or that x is equal to $y^{\frac{n}{m}}$.

6thly. *Powers are reducible to more ſimple powers, as often as their fractional indexes are reducible to more ſimple fractions.* Thus the ſquare-ſquare root of x^{2} is the ſame with the ſquare root of x, becauſe $x^{\frac{2}{4}}=x^{\frac{1}{2}}$.

7thly. *If the index of any power be an improper fraction, and that fraction be reduced into a whole number and a fraction, the power will hereby be reſolved into two factors, whereof one will have the whole number for it's index, and the other the fractional part.* Thus $\frac{5}{2}=2+\frac{1}{2}$, and therefore $x^{\frac{5}{2}}=x^{2}\times x^{\frac{1}{2}}$; that is, the ſquare root of x^{5} is equal to xx multiplied into the ſquare root of x.

8thly. *Surd powers may be reduced to the ſame root by a reduction of their fractional indexes to the ſame denomination, and that, whether they be powers of the ſame quantity or not.* Thus $x^{\frac{1}{2}}$ and $y^{\frac{1}{3}}$ are the ſame as $x^{\frac{3}{6}}$ and $y^{\frac{2}{6}}$; that is, the ſquare root of x, and the cube root of y are the ſame as the ſixth root of x^{3} and the ſixth root of y^{2}: and thus may ſurds of different

different roots be compared together without any extraction of those roots. As for instance, if any one should ask me, which of these two quantities is the greater, the square root of 2 or the cube root of 3? I should answer, the cube root of 3; for the square root of 2 or $2^{\frac{1}{2}}$, or $2^{\frac{3}{6}}$, is equal to $8^{\frac{1}{6}}$; but the cube root of 3, or $3^{\frac{1}{3}}$, or $3^{\frac{2}{6}}$, is equal to $9^{\frac{1}{6}}$; and $9^{\frac{1}{6}}$ is greater than $8^{\frac{1}{6}}$.

9thly. *That the addition and subtraction of indexes answers to the multiplication and division of the powers to which they belong, holds equally true in fractional indexes, as in integral ones.* Thus $\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$, and $x^{\frac{1}{2}}\times x^{\frac{1}{3}}=x^{\frac{5}{6}}$, which I thus demonstrate. Let $y^6=x$; then by the fifth observation we shall have $y=x^{\frac{1}{6}}$, $y^3=x^{\frac{3}{6}}$ or $x^{\frac{1}{2}}$, $y^2=x^{\frac{2}{6}}$ or $x^{\frac{1}{3}}$, and $y^5=x^{\frac{5}{6}}$: but $y^3\times y^2$ is equal to y^5 by the third observation; therefore $x^{\frac{1}{2}}$ multiplied into $x^{\frac{1}{3}}$ gives $x^{\frac{5}{6}}$. After the same manner, since $\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$, it may be demonstrated that $x^{\frac{1}{2}}$ divided by $x^{\frac{1}{3}}$ will give $x^{\frac{1}{6}}$; for y^3 divided by y^2 gives y, which is equal to $x^{\frac{1}{6}}$; and the demonstrations will be the same in all other cases.

Of Newton's *theorem for the evolution of a binomial, or rather the powers of a binomial, into serieses finite or infinite, as the nature of such power will admit.*

380. This theorem is so very useful in almost all the parts of Mathematics, and more especially in the sublimer parts of Geometry, that I hope I shall be excused if I insist somewhat the longer upon it.

$\overline{1+x}^{m}$ *signifies that power of the binomial* $1+x$ *whose index* is m. Thus $\overline{1+x}^{0}=1$, for the same reason that $x^0=1$: thus $\overline{1+x}^{1}=1+x$, for the same reason that $x^1=x$: and if a continual multiplication be made by $1+x$, beginning with the first multiplicand $1+x$, we shall form other powers of the binomial $1+x$, as follows:

$$\overline{1+x}^{0}=1.$$

$$\overline{1+x}^{1}=1+x.$$

$$\overline{1+x}^{2}=1+2x+xx.$$

$$\overline{1+x}^{3}=1+3x+3xx+x^3.$$

$$\overline{1+x}^{4}=1+4x+6xx+4x^3+x^4.$$

$$\overline{1+x}^{5}=1+5x+10xx+10x^3+5x^4+x^5.$$

$$\overline{1+x}^{6}=1+6x+15xx+20x^3+15x^4+6x^5+x^6.$$

N. B.

N. B. *The numeral coefficients of the terms of each series are called the unciæ of the power to which the series belongs.* Thus $\overline{1+x}^6 = 1+6x+15xx+20x^3+15x^4+6x^5+1x^6$, and the numeral coefficients 1, 6, 15, 20, 15, 6, 1 are called the *unciæ* of the sixth power.

381. Having thus by continual multiplication obtained the sixth power of the binomial $1+x$, consisting of a competent number of terms to form an induction from, let us in the next place inquire what relation these terms have one to another (if they have any visible one,) or how they may be formed one from another, that so (if possible) we may be able *per saltum*, without the help of intermediate powers, to form a series exhibiting any given power whatever of the binomial root $1+x$. To do this, I am to acquaint the reader that a great many serieses of almost all kinds, are formed by a continual addition, or a continual multiplication of the terms of other serieses more simple than those generated by them; and therefore whenever a series is proposed, the law or *genesis* whereof is not immediately perceived, it will not be amiss to inquire by what continual additions or multiplications it's terms are produced: the additions are easily known by subtracting every term from the next following, and the multiplications (which is our case at present) by dividing every term by the term next before it: thus the series $1+a+b+c+d$ &c is generated by a continual multiplication of the terms $\frac{a}{1}, \frac{b}{a}, \frac{c}{b}, \frac{d}{c}$ &c; for $1\times\frac{a}{1}$ gives a, and $a\times\frac{b}{a}$ gives b, and $b\times\frac{c}{b}$ gives c &c. Now if we apply this to the series exhibiting the sixth power of the binomial $1+x$, to wit, $1+6x+15x^2+20x^3+15x^4+6x^5+1x^6$, we shall find the terms of this series to be produced by a continual multiplication of the following terms; 1st $\frac{6x}{1}$, 2dly $\frac{15xx}{6x}$ or $\frac{5x}{2}$, 3dly $\frac{20x^3}{15x^2}$ or $\frac{4x}{3}$, 4thly $\frac{15x^4}{20x^3}$ or $\frac{3x}{4}$, 5thly $\frac{6x^5}{15x^4}$ or $\frac{2x}{5}$, lastly $\frac{1x^6}{6x^5}$ or $\frac{1x}{6}$: so that the series whose terms continually multiplied give the sixth power of the binomial $\overline{1+x}$, is $\frac{6x}{1}+\frac{5x}{2}+\frac{4x}{3}+\frac{3x}{4}+\frac{2x}{5}+\frac{1x}{6}$: the regularity of this series is apparent enough at first sight, the coefficients of the numerators constantly decreasing, and the denominators constantly increasing by unity; insomuch that none can doubt the obtaining of a like regularity in all the other powers; at least if he does, he may confirm this induction by trying as many other powers as he shall think fit; where he will always find that if m be the index of any power, the fractions by whose continual multiplication all the terms of that power after the first

first are generated, will be $\frac{m}{1}x$, $\frac{m-1}{2}x$, $\frac{m-2}{3}x$, $\frac{m-3}{4}x$, $\frac{m-4}{5}x$, $\frac{m-5}{6}x$ &c. But from the formation of these powers one from another by a continual multiplication into $1+x$ it is easy to see, that unity or 1 will always be the first term of every power; therefore *If we make this first term* $1=A$, $\frac{m}{1}Ax=B$, $\frac{m-1}{2}Bx=C$, $\frac{m-2}{3}Cx=D$, $\frac{m-3}{4}Dx=E$, $\frac{m-4}{5}Ex=F$, $\frac{m-5}{6}Fx=G$ &c, *we shall have* $\overline{1+x}^m = A+B+C+D+E+F+G$ &c: *or more elegantly (according to* Newton's *manner of expressing such serieses) thus,* $\overline{1+x}^m = 1+\frac{m}{1}Ax + \frac{m-1}{2}Bx + \frac{m-2}{3}Cx + \frac{m-3}{4}Dx + \frac{m-4}{5}Ex + \frac{m-5}{6}Fx$ &c, *where the capital letters* A, B, C &c *represent the terms of the series as they rise; that is,* A *represents the first term* 1, B *signifies the second term* $\frac{m}{1}Ax$, C *signifies the third term* $\frac{m-1}{2}Bx$ &c. Examples of this will be given in the following article.

Examples to the foregoing theorem.

382. 1st. Let it be required to raise at once the binomial $1+x$ to the seventh power. Here $\frac{m}{1}=\frac{7}{1}$, $\frac{m-1}{2}=\frac{6}{2}$, $\frac{m-2}{3}=\frac{5}{3}$, &c; therefore $\overline{1+x}^7 = 1+\frac{7}{1}Ax+\frac{6}{2}Bx+\frac{5}{3}Cx+\frac{4}{4}Dx+\frac{3}{5}Ex+\frac{2}{6}Fx+\frac{1}{7}Gx$. Having thus laid out the series, by expressing $\frac{m}{1}$, $\frac{m-1}{2}$, $\frac{m-2}{3}$ &c in numbers, the terms must next be computed thus; 1st $1=A$, 2dly $\frac{7}{1}Ax = \frac{7}{1}\times 1x = 7x = B$, 3dly $\frac{6}{2}Bx = \frac{6}{2}\times 7xx = 21xx = C$, 4thly $\frac{5}{3}Cx = \frac{5}{3}\times 21x^3 = 35x^3 = D$, 5thly $\frac{4}{4}Dx = \frac{4}{4}\times 35x^4 = 35x^4 = E$, 6thly $\frac{3}{5}Ex = \frac{3}{5}\times 35x^5 = 21x^5 = F$, 7thly $\frac{2}{6}Fx = \frac{2}{6}\times 21x^6 = 7x^6 = G$, 8thly $\frac{1}{7}Gx = \frac{1}{7}\times 7x^7 = x^7 = H$; therefore $\overline{1+x}^7 = 1+7x+21x^2+35x^3+35x^4+21x^5+7x^6+x^7$.

2d. $\overline{1+x}^8 = 1+\frac{8}{1}Ax+\frac{7}{2}Bx+\frac{6}{3}Cx+\frac{5}{4}Dx+\frac{4}{5}Ex+\frac{3}{6}Fx+\frac{2}{7}Gx+\frac{1}{8}Hx = 1+8x+28x^2+56x^3+70x^4+56x^5+28x^6+8x^7+x^8$.

3d.

3d. $\overline{1-x}|^5 = 1 + \frac{5}{1} \times -Ax + \frac{4}{2} \times -Bx + \frac{3}{3} \times -Cx + \frac{2}{4} \times -Dx + \frac{1}{5} \times -Ex$, where the sign $+$ signifies no more than that the subsequent terms are to be added to the foregoing, whether they be affirmative or negative: or it might be expressed thus; $\overline{1-x}|^5 = 1 - \frac{5}{1}Ax - \frac{4}{2}Bx - \frac{3}{3}Cx - \frac{2}{4}Dx - \frac{1}{5}Ex$. The terms may be computed thus; 1st $1 = A$, 2dly $-\frac{5}{1}Ax = -\frac{5}{1} \times 1x = -5x = B$, 3dly $-\frac{4}{2}Bx = -\frac{4}{2} \times -5xx = +10xx = C$, 4thly $-\frac{3}{3}Cx = -\frac{3}{3} \times 10x^3 = -10x^3 = D$, 5thly $-\frac{2}{4}Dx = -\frac{2}{4} \times -10x^4 = +5x^4 = E$, 6thly $-\frac{1}{5}Ex = -\frac{1}{5} \times 5x^5 = -x^5 = F$; therefore $\overline{1-x}|^5 = 1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5$, the terms being alternately affirmative and negative.

4th. $\overline{1-x}|^4 = 1 - 4x + 6x^2 - 4x^3 + x^4$.

From the two last examples it appears, that if any power of the binomial $1+x$ *be obtained, and then the signs of all those terms wherein the odd powers of* x *are concerned be changed, you will have the same power of the binomial* $1-x$. Thus $\overline{1+x}|^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$: change the signs of the terms $+5x$, $+10x^3$, $+x^5$, and you will have $\overline{1-x}|^5 = 1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5$, as before. The reason of this is plain; for if x be negative, every odd power of x will also be negative, whereas every even power will be affirmative as much as if x itself was affirmative: thus $-x \times -x = +x^2$, and $+x^2 \times -x = -x^3$, and $-x^3 \times -x = +x^4$, and $+x^4 \times -x = -x^5$, &c.

383. *If any power of a binomial be to be multiplied by any given number as* n, *this may be effected two ways; to wit, either by multiplying every term whereof that power consists by* n, *or else by multiplying only the first term, which will always be known, by* n, *and then (calling that product* A) *deriving all the other terms from it as before.* As for example; $\overline{1-x}|^4 \times n$ may either be expressed thus, $n - 4nx + 6nx^2 - 4nx^3 + nx^4$; or thus, $n - \frac{4}{1}Ax - \frac{3}{2}Bx - \frac{2}{3}Cx - \frac{1}{4}Dx$: for the terms of this latter series, when computed, will be found the same with those of the former: thus the first term will be $n = A$, the second will be $-\frac{4}{1}Ax = -\frac{4}{1} \times nx = -4nx$, and so of the rest.

384. *By the help of this last article we may express by a series any given power of any binomial whatever, as* $\overline{p+q}|^m$, *by considering the binomial* $p+q$ *as the product of two factors, to wit,* $1+\frac{q}{p}$ *and* p: for if $p+q = \overline{1+\frac{q}{p}} \times p$, we shall have $\overline{p+q}|^m = \overline{1+\frac{q}{p}}|^m \times p^m$; but the first

term of $\overline{1+\frac{q}{p}}^{m}$ when thrown into a series will be 1: multiply then this first term 1 into p^m; make the product $p^m=A$, as in the last article, and derive all the other terms from A as above directed, putting $\frac{q}{p}$ for x, and you will have $\overline{p+q}^{m}=p^m+\frac{m}{1}\frac{Aq}{p}+\frac{m-1}{2}\frac{Bq}{p}+\frac{m-2}{3}\frac{Cq}{p}$ &c. Thus $\overline{p+q}^{3}=p^3+\frac{3}{1}\frac{Aq}{p}+\frac{2}{2}\frac{Bq}{p}+\frac{1}{3}\frac{Cq}{p}=p^3+3p^2q+3pq^2+q^3$: put 1 instead of q, and you will have $\overline{p+1}^{3}=p^3+3pp+3p+1$ which is, and ought to be the same with $\overline{1+p}^{3}$; it is only beginning at the other end of the series. And this is the reason that *In expanding the binomial* $\overline{\mathrm{p}+\mathrm{q}}^{m}$, *the* unciæ *of any two terms equally distant from the extremes are the same.*

385. *From what has been here laid down it may be observed, that if* m, *the index of the power to which the binomial* $1+\mathrm{x}$ *is to be raised, be integral and affirmative, the series exhibiting that power will at last break off, and so consist but of a finite number of terms, otherwise the series will run on* ad infinitum: for if m be integral and affirmative, the progression of numerators m, $m-1$, $m-2$, $m-3$ &c must, if continued, pass through $m-m$ or 0, which will make one of the terms A, B, C, D, E, F, G &c equal to nothing; and if any one of these terms be equal to nothing, it will necessarily destroy all those that follow it, and so the series will be interrupted. As for instance, the series exhibiting the third power of $1+x$ will, if regularly continued, stand thus, $1+\frac{3}{1}Ax+\frac{2}{2}Bx+\frac{1}{3}Cx+\frac{0}{4}Dx-\frac{1}{5}Ex-\frac{2}{6}Fx$ &c; but of this series the term E, which is equal to $\frac{0}{4}Dx$, will be nothing; and if E be nothing, then F, which is equal to $-\frac{1}{5}Ex$ will also be nothing; and if F be nothing, then G, which is equal to $-\frac{2}{6}Fx$ will also be equal to nothing; so that the first four terms will be real, and then the series will break off. If m be negative, $m-1$ will be more negative, and $m-2$ still more negative; so that in this case the progression m, $m-1$, $m-2$ &c cannot pass through nothing; and therefore the series A, B, C, D &c must be infinite, I mean as to the number of terms. If m be not integral, but a fraction, as if $m=\frac{7}{2}$, we shall have $m-1=\frac{5}{2}$, $m-2=\frac{3}{2}$, $m-3=\frac{1}{2}$, $m-4=-\frac{1}{2}$, $m-5=-\frac{3}{2}$ &c; therefore in this case the progression m, $m-1$, $m-2$, $m-3$ &c may be said to pass by nothing, but not through it; and therefore here also the series will be infinite. But of these infinite serieses more will be said in the following articles.

386.

386. *As* $+1$ *is the index of the simple power of* $1+x$, *so* -1 *will be the index of* $\frac{1}{1+x}$: for it has been shewn already that whatever number is the index of any power, it's negative will be the index of the reciprocal of that power, or of unity divided by it. Let it then be required to throw the fraction $\frac{1}{1+x}$ or $\overline{1+x}^{-1}$ into an infinite series: here $\frac{m}{1}=-1$, $\frac{m-1}{2}=-\frac{2}{2}=-1$, $\frac{m-2}{3}=-\frac{3}{3}=-1$ &c; therefore in this case every coefficient will be -1; and so we shall have $\overline{1+x}^{-1}$ $=1-Ax-Bx-Cx-Dx-Ex-Fx$ &c *ad infinitum*, $=1-x+x^2-x^3+x^4-x^5+x^6-$ &c *ad infinitum*. In like manner we have $\overline{1-x}^{-1}$ $1+x+x^2+x^3+x^4+x^5+x^6$ &c *ad infinitum*.

387. Again, let it be required to throw the fraction $\frac{Ap}{p-q}$ into a series. Here we may consider $\frac{Ap}{p-q}$ as Ap multiplied into $\frac{1}{p-q}$ or as Ap multiplied into $\overline{p-q}^{-1}$: but the first term of $\overline{p-q}^{-1}$, when thrown into a series, is p^{-1} or $\frac{1}{p}$; see art. 384: multiply this first term $\frac{1}{p}$ into the common multiplicator Ap, and the product A will be the first term of the series: whence $\frac{Ap}{p-q}=A+\frac{Aq}{p}+\frac{Bq}{p}+\frac{Cq}{p}+\frac{Dq}{p}$ &c *ad infinitum*, $=A+\frac{Aq}{p}+\frac{Aq^2}{p^2}+\frac{Aq^3}{p^3}+\frac{Aq^4}{p^4}$ &c, which is an infinite series whose terms are in continual geometrical proportion, the common ratio being that of p to q; that is, every term being to the next following as p to q; therefore the less the quantity q is in comparison of the other quantity p, the less will every succeeding term be in comparison of the next before it, and the faster will the series converge, that is, the nearer will any number of terms taken from the beginning of the series, approach towards the original fraction $\frac{Ap}{p-q}$, provided that there be taken always the same number of terms: whence it follows again, that *The less* q *is in comparison of* p, *the fewer terms need be taken to represent the whole series to the same degree of exactness; and this is not only true in the present case, but in the case of all other serieses arising from a binomial root* p+q *or* p−q.

The series, that it may converge, must always take it's rise from the greater part of the binomial. As if p be greater than q in the binomial $p \pm q$, the series must always be begun with the quantity p: for let $\overline{p+q}^m$ be thrown into a series, and we shall have $\overline{p+q}^m = p^m + \frac{m}{1}\frac{Aq}{p} + \frac{m-1}{2}\frac{Bq}{p} + \frac{m-2}{3}\frac{Cq}{p}$ &c, as in art. 384. Let q be supposed equal to nothing, or at least infinitely small in comparison of p; and then it is easy to see, that every succeeding term of this series will be infinitely less than that next before it, that therefore all the terms after the first may be looked upon as evanescent, and consequently that the whole series will be comprehended in the first term, as it ought to be: for if $q=0$, then $\overline{p+q}^m = p^m$; therefore if q be nearly equal to nothing, a few of the initial terms of the series will be nearly equal to the whole, and in cases where the utmost exactness is not required, may be taken for the whole.

But to return; since the fraction $\frac{Ap}{p-q}$ was found equal to the infinite series $A + \frac{Aq}{p} + \frac{Aq^2}{p^2} + \frac{Aq^3}{p^3} + \frac{Aq^4}{p^4}$ &c when q is less than p, the terms of which series decrease in continual proportion, (whereof more will be said in the next article,) it follows *e converso*, that *A series of quantities decreasing in a continual geometrical proportion, such as* $A + \frac{Aq}{p} + \frac{Aq^2}{p^2}$ *&c, will, though infinitely continued, be but equal to a finite quantity; which finite quantity may be had by multiplying* A, *the first and greatest term of the series, into* p, *the antecedent of the common ratio, and then dividing the product by* p—q, *the excess of the antecedent above the consequent*: of this take an example or two in numbers.

Let $p=2$, $q=1$, and $A=1$; then will $\frac{Ap}{p-q}=2$, and the series will be $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$ &c; and therefore *e converso*, this series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$ &c, though infinitely continued, will be but equal to 2, which I thus further confirm. The first term 1 equals $2-1$ or $2-\frac{2}{2}$; therefore $1+\frac{1}{2}=2-\frac{1}{2}=2-\frac{2}{4}$; therefore $1+\frac{1}{2}+\frac{1}{4}=2-\frac{1}{4}$ or $2-\frac{2}{8}$; therefore $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=2-\frac{1}{8}$ or $2-\frac{2}{16}$; therefore $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}=2-\frac{1}{16}$ &c. Whence it appears that this series can never exceed the number 2, or even reach it unless an infinite number of additions could be made. Let $p=10$, $q=1$, and $A=1$, and

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we shall have the infinite series $1+\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\frac{1}{10000}$ &c $=$ $\frac{10}{9}$ or $1+\frac{1}{9}$; and this will be further confirmed by reducing the fraction $\frac{1}{9}$ into decimals, which decimals will be $.11111$ &c *ad infinitum.*

If the fraction $\frac{Ap}{p+q}$ be thrown into a series, the series will be $A-\frac{Aq}{p}+\frac{Aq^2}{p^2}-\frac{Aq^3}{p^3}+\frac{Aq^4}{p^4}-$ &c *ad infinitum.* Let $p=2$, $q=1$, and $A=1$, as in the first instance, and we shall have $\frac{Ap}{p+q}=\frac{2}{3}$, and the series will be $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}$ &c: whence *e converso*, the infinite series $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}$ &c will be equal to the fraction $\frac{2}{3}$, which I thus further confirm. The first term 1 equals $\frac{2}{3}+\frac{1}{3}$; therefore $1-\frac{1}{2}=\frac{2}{3}-\frac{1}{6}$; therefore $1-\frac{1}{2}+\frac{1}{4}=\frac{2}{3}+\frac{1}{12}$; therefore $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}=\frac{2}{3}-\frac{1}{24}$; therefore $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}=\frac{2}{3}+\frac{1}{48}$ &c.

Let it be required to throw this fraction $\frac{1}{1+2x+xx}$ or $\overline{1+x}^{-2}$ into an infinite series. Here $\frac{m}{1}$, $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$ &c $=\frac{-2}{1}$, $\frac{-3}{2}$, $\frac{-4}{3}$, $\frac{-5}{4}$, $\frac{-6}{5}$ &c: whence $\overline{1+x}^{-2}=1-\frac{2}{1}Ax-\frac{3}{2}Bx-\frac{4}{3}Cx-\frac{5}{4}Dx-\frac{6}{5}Ex$ &c $=1-2x+3x^2-4x^3+5x^4-6x^5$ &c *ad infinitum.*

388. *As all powers of a binomial whose exponents are integral and affirmative may be obtained by continual multiplication, so all those whose exponents are integral and negative may be had by continual division.* Thus $\overline{1+x}^{-1}$ or $\frac{1}{1+x}$ was in the last article but one, according to *Newton*'s theorem, $1-x+x^2-x^3+x^4-x^5+x^6$ &c; and the same will be the quotient if the fraction $\frac{1}{1+x}$ be thrown into an infinite series by an actual division of 1 by $1+x$ according to the common rules of division, supplying the deficient places of the dividend with stars, as was done in art. 14: if this quotient be again divided by $1+x$, we shall have $\frac{1}{\overline{1+x}\times\overline{1+x}}$ or $\overline{1+x}^{-2}=1-2x+3x^2-4x^3+5x^4-6x^5$ &c, as in the last article; and so on.

But here a question may arise which perhaps would puzzle even an ingenious reader (that is not very conversant in these affairs) to answer

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to his own satisfaction: it is this; since these serieses are obtained by division, how comes it to pass that, though infinitely continued, they do not always exhibit the true quotient? as in many cases it is certain they do not: for if 1 be divided by $1+x$, the quotient will be $1-x+x^2-x^3+x^4$ &c as above. Let us now suppose x equal to 1, and the true quotient of 1 divided by $1+x$ is certainly $\frac{1}{2}$; but the series which should exhibit this quotient will, in this case, be $1-1+1-1+1-1$ &c *ad infinitum*, whereof if an even number of terms be taken, the sum will be 0, if an odd number, the sum will be 1; so that the quotient exhibited by this series will always be too much or too little, and never can be equal to $\frac{1}{2}$.

To this I answer; if a division of 1 by $1+x$ be made for one term in the quotient only, the quotient will be 1, and the remainder will be $-x$; if the division be carried on to two terms only, the quotient will be $1-x$, and the remainder $+xx$; if to three terms, the quotient will be $1-x+xx$, and the remainder $-x^3$; if to four terms, the quotient will be $1-x+xx-x^3$, and the remainder $+x^4$, and so on *ad infinitum*. Now if x be less than 1, xx will be less than x, and so on, in which case the remainders will lessen upon our hands; and if the division be infinitely continued, the remainders will become evanescent quantities, in which case the dividend will be just exhausted, and the series $1-x+x^2$ &c will be the true quotient. If x be equal to 1, every power of x, as x^2, x^3 &c, will be equal to 1; and therefore in this case the remainder will always be $+1$ or -1 even though the division be continued *ad infinitum*; but wherever there is a remainder, the quotient will not be exact, but will always be too little or too much, according as the remainder is affirmative or negative; therefore in this case the series $1-x+x^2$ &c, even though infinitely continued, will not exhibit the true quotient. If x be greater than 1, xx will be greater than x, and x^3 greater than xx, and so on; therefore in this case the remainders will be so far from lessening, that they will grow constantly greater and greater; and if the division be infinitely continued, the remainders will be infinitely great; and therefore the series $1-x+x^2$ &c, if infinitely continued, will be infinitely wide of the true quotient: these are what we call diverging serieses: but if x be greater than 1, and the division be begun by x instead of 1, according to the directions given in the last article, the series will converge to the true quotient faster or slower according as the excess of x above 1 is greater or less; for in this case, the quotient of 1 divided by $x+1$ will be $\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3}-\frac{1}{x^4}+$ &c *ad infinitum*.

In common division it is usual, where there is a remainder at last, to correct the quotient by a supplemental fraction whose numerator is the remainder

remainder and whofe denominator is the divifor: if the fame be done here, the quotient will always be exact, at what end foever of the divifor the divifion commences, or wherever it breaks off: thus if 1 be divided by $1+x$, the quotient will be $\frac{1}{1+x}$, or $1-\frac{x}{1+x}$, or $1-x+\frac{x^2}{1+x}$, or $1-x+x^2-\frac{x^3}{1+x}$, and fo on; that is, if $x=1$, the quotient will be $\frac{1}{2}$, or $1-\frac{1}{2}$, or $1-1+\frac{1}{2}$, or $1-1+1-\frac{1}{2}$, and fo on; all which give one and the fame fraction $\frac{1}{2}$.

389. Let it now be required to extract the fquare root of $1+x$ by *Newton*'s theorem, as before was done the common way; fee art. 19 cafe 1. Now $\sqrt{1+x}$, according to *Newton*'s expreffion, is $\overline{1+x}^{\frac{1}{2}}$; therefore $\frac{m}{1}$, $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, $\frac{m-5}{6}$ in this cafe, are equal to $\frac{1}{2}$, $-\frac{1}{4}$, $-\frac{3}{6}$, $-\frac{5}{8}$, $-\frac{7}{10}$, $-\frac{9}{12}$ &c which is a regular feries of fractions whofe numerators decreafe and denominators increafe by 2; fo that $\overline{1+x}^{\frac{1}{2}}=1+\frac{1}{2}Ax-\frac{1}{4}Bx-\frac{3}{6}Cx-\frac{5}{8}Dx$ &c $=1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\frac{5}{128}x^4$ &c, as in the cafe above quoted. This may eafily be tried by multiplication, as in the fame cafe: for if this root when multiplied into itfelf produces $1+x$, it will be a fufficient proof that the root was truly affigned.

If x be fuppofed greater than 1, we muft then have begun the computation with x, and the firft term of the feries would have been $x^{\frac{1}{2}}$ or $\sqrt{x}$, and fo we fhould have had $\overline{x+1}^{\frac{1}{2}}=\sqrt{x}+\frac{1}{2}\frac{\sqrt{x}}{x}-\frac{1}{8}\frac{\sqrt{x}}{x^2}+\frac{1}{16}\cdot\frac{\sqrt{x}}{x^3}$ &c, or $\sqrt{x}\times 1+\frac{\frac{1}{2}}{x}-\frac{\frac{1}{8}}{x^2}+\frac{\frac{1}{16}}{x^3}$ &c.

If it be required to expand this fraction $\frac{1}{\sqrt{1-xx}}$ or $\overline{1-xx}^{-\frac{1}{2}}$ into an infinite feries, it will ftand thus; $\overline{1-xx}^{-\frac{1}{2}}=1+\frac{1}{2}Axx+\frac{3}{4}Bxx+\frac{5}{6}Cxx+\frac{7}{8}Dxx+\frac{9}{10}Exx$ &c $=1+\frac{1}{2}x^2+\frac{3}{8}x^4+\frac{5}{16}x^6+\frac{35}{128}x^8+\frac{63}{256}x^{10}$ &c.

Laftly, let it be required to throw this quantity $\overline{1+x}^{\frac{m}{n}}$ into an infinite feries: here inftead of m, $\frac{m-1}{2}$, $\frac{m-2}{3}$, &c we muft ufe $\frac{m}{n}$, $\frac{\frac{m}{n}-1}{2}$, $\frac{\frac{m}{n}-2}{3}$ &c; or $\frac{m}{n}$, $\frac{m-n}{2n}$, $\frac{m-2n}{3n}$ &c, therefore $\overline{1+x}^{\frac{m}{n}}=1+$

$+\frac{m}{n}Ax+\frac{m-n}{2n}Bx+\frac{m-2n}{3n}Cx+\frac{m-3n}{4n}Dx$ &c; where the ſigns + only ſignify that theſe terms are to be added together according to the rules of addition, whether they happen to be affirmative or negative. If the index $\frac{m}{n}$ be left undetermined, and but a few terms of the ſeries be required, it will then be as well to write it thus, $\overline{1+x}|^{\frac{m}{n}}=1+\frac{m}{n}x+\frac{m}{n}\times\frac{m-n}{2n}x^2+\frac{m}{n}\times\frac{m-n}{2n}\times\frac{m-2n}{3n}x^3$ &c.

I might have demonſtrated this theorem for finding the *unciæ* of an expanded binomial various other ways; but I ſuppoſe by this time the reader has enough of it, at leaſt till he ſees it's uſe: therefore I ſhall proceed in the next place to apply it to the computation of logarithms.

PART II.

Of logarithms, their uſe, and the beſt methods of computing them.

The definition of logarithms, and conſectaries drawn from it.

390. LOGARITHMS *are a ſet of artificial numbers placed over againſt the natural ones, uſually from* 1 *to* 100000, *and ſo contrived that their addition anſwers to the multiplication of the natural numbers to which they belong*; *that is, if any two numbers be multiplied together, and ſo produce a third, their logarithms being added together will conſtitute the logarithm of that third.*

Thus 0.3010300, the common logarithm of 2, added to
0.4771213, the logarithm of 3, gives
0.7781513, the logarithm of 6, becauſe 6 is the product of 2 and 3 multiplied together.

From this definition it follows firſt, *That in any ſyſtem or table of logarithms whatever, the logarithm of unity or* 1 *will be nothing:* for as 1 neither increaſes nor diminiſhes the number multiplied by it, ſo neither will it's logarithm either increaſe or diminiſh the logarithm to which it is added; and therefore the logarithm of 1 muſt be nothing.

2dly.

2dly. *For a like reason, the logarithm of a proper fraction will always be negative*: for such a fraction always diminishes the number multiplied by it, and therefore it's logarithm will always diminish the logarithm to which it is added.

3dly. *This property of logarithms, whereby they are defined as above, affords us no small compendium in multiplication*: for whenever one number is to be multiplied by another, it is but taking out their logarithms, and adding them together, and their sum will be a third logarithm whose natural number being taken out of the tables will be the product required.

4thly. *The subtraction of logarithms answers to the division of the natural numbers to which they belong*; that is, whenever one number is to be divided by another, it is but subtracting the logarithm of the divisor from the logarithm of the dividend, and the remainder will be the logarithm of the quotient: and thus by the help of logarithms may the operation of division be performed by meer subtraction as that of multiplication was by addition. *Hence, as every fraction is nothing else but the quotient of the numerator divided by the denominator, it's logarithm will be found by subtracting the logarithm of the denominator from the logarithm of the numerator.* To demonstrate this, to wit, that the logarithm of the divisor subtracted from the logarithm of the dividend will leave the logarithm of the quotient, let the number A be divided by the number B, and let the quotient be the number C, and let the logarithms of the numbers A, B and C be a, b and c respectively; I say then that $a-b$ will be equal to c: for since by the supposition $\frac{A}{B}=C$, we shall have $A=BC$, and $a=b+c$ by the definition; whence $a-b=c$.

5thly. *As every fourth proportional is found by multiplying the second and third numbers together, and dividing the product by the first, so the logarithm of every such fourth proportional will be found by adding the logarithms of the second and third numbers together, and subtracting from the sum the logarithm of the first.* This renders all operations by the rule of proportion very compendious and easy; especially after the practitioner has pretty well inured himself to take out of the table logarithms to his numbers, and numbers to his logarithms: but this compendium is chiefly useful in Trigonometry, both plain and spherical, where every thing he wants is put down ready to his hands.

6thly. *If* A *be any number whose logarithm is* a, *then the logarithm of* A^2 *will be* 2a, *that of* A^3, 3a *&c, that of* $\frac{1}{A}$, —a, *that of* $\frac{1}{A^2}$, —2a *&c. And universally, the logarithm of* A^m *will be* a×m, *and that, whether the index* m *be integral or fractional, affirmative or negative: on the other hand,*

if q *be the logarithm of any power of* A, *as of* A^m, *then* $\frac{q}{m}$ *will be the logarithm of* A. The reafon of all this is plain; for as A^2 is the product of A multiplied into itfelf, fo it's logarithm will be the logarithm of A added to itfelf or doubled, that is $2a$; and fo of the higher powers. Again, as $\frac{1}{A}$ is the quotient of unity divided by A, it's logarithm will be found by fubtracting a, the logarithm of A, from 0, the logarithm of 1, which gives $-a$; and fo of the lower powers. Laftly, as $\sqrt{A}$, when multiplied into itfelf, produces A, fo it's logarithm, when added to itfelf, ought to make a; therefore the logarithm of $\sqrt{A}$ will be $\frac{1}{2}a$; and fo of all the other fractional powers. Here then again we have another inftance of the very great ufefulnefs of a good table of logarithms, to wit, in raifing a number to any given power, or in extracting any given root out of it, all which is performed with equal facility, only by multiplying it's logarithm by the index of the given power, or dividing it by the index of the given root; as doubling it for the fquare, tripling it for the cube &c; halving it for the fquare root, trifecting it for the cube root &c: this, I fay, cannot but be very ufeful in a great many cafes, and more efpecially in Anatocifm, where we have fometimes occafion to extract even the three hundred fixty fifth root of a number, as at other times to raife it to the three hundred fixty fifth power, fcarce poffible to be performed any other way; to fay nothing of the innumerable miftakes that in fo long and laborious a calculation would be almoft unavoidable, all which are prevented by the ufe of logarithms. It cannot indeed be expected that entire powers, and much lefs entire roots fhould be gained this way; but it will be eafy in moft cafes to obtain as many terms as can be of any ufe to us.

7thly. *If any fet of numbers, as* A, B, C, D *be in continual geometrical proportion, their logarithms, which we fhall call* a, b, c, d, *will be in arithmetical progreffion:* for fince by the fuppofition A is to B as B is to C as C is to D, that is, fince $\frac{B}{A}=\frac{C}{B}=\frac{D}{C}$, we fhall have $b-a=c-b=d-c$ by the fourth confectary; therefore a, b, c, d are in arithmetical progreffion. *Q. E. D.*

8thly. *From this laft confectary it will be eafy, having two numbers given, to find as many mean proportionals as we pleafe between them.* Let the given numbers be A and F, and let it be required to find four mean proportionals between them, which we fhall call B, C, D, E, fo that A, B, C, D, E, F may be in continual geometrical proportion. Here then it is evident from the laft confectary, that as thefe numbers are

are in continual geometrical proportion, their logarithms, which we ſhall call a, b, c, d, e, f, will be in arithmetical progreſſion, whereof the extremes a and f are known, as being the logarithms of the known numbers A and F, and the intermediates may be found thus. Put x for the common difference of this arithmetic progreſſion; then will $a+x=b$, $a+2x=c$, $a+3x=d$, $a+4x=e$, $a+5x=f$; whence $x=\frac{f-a}{5}$; whence $a+x$ or $b=a+\frac{f-a}{5}=\frac{4a+f}{5}$, $a+2x$ or $c=\frac{3a+2f}{5}$, $a+3x$ or $d=\frac{2a+3f}{5}$, $a+4x$ or $e=\frac{a+4f}{5}$; ſo that the logarithms of the four mean proportionals ſought are $\frac{4a+f}{5}$, $\frac{3a+2f}{5}$, $\frac{2a+3f}{5}$, $\frac{a+4f}{5}$: take then the natural numbers B, C, D, E of theſe logarithms, and they will be the mean proportionals required. *Q. E. I.*

Logarithms the meaſures of ratios.

391. *Logarithms are ſo called from their being the arithmetical or numeral exponents of ratios:* for if unity be made the common conſequent of all ratios, or the common ſtandard to which all other numbers are to be referred, then every logarithm will be the numeral exponent of the ratio of it's natural number to unity. As for inſtance, the ratio of 81 to 1 actually contains within itſelf theſe four ratios, to wit, the ratio of 81 to 27, that of 27 to 9, that of 9 to 3, and that of 3 to 1, (ſee art. 293;) all which ratios are equal to one another, and to the ratio of 3 to 1; therefore the ratio of 81 to 1 is ſaid to be four times as big as the ratio of 3 to 1, (ſee art. 294:) and hence it is that the logarithm of 81 is four times as big as the logarithm of 3. Again, the ratio of 24 to 1 contains, and may be reſolved into theſe three ratios, to wit, the ratio of 24 to 12, that of 12 to 4, and that of 4 to 1; the firſt of theſe ratios, to wit, the ratio of 24 to 12, is the ſame with that of 2 to 1; the ſecond, to wit, the ratio of 12 to 4 is the ſame with that of 3 to 1; and therefore the ratio of 24 to 1 is equal to the ratios of 2 to 1, 3 to 1, and 4 to 1 put together; and hence it is that the logarithm of 24 is equal to the logarithms of 2, 3 and 4 put together: *And univerſally, the magnitude of the ratio of* A *to* 1 *is to the magnitude of the ratio of* B *to* 1 *as the logarithm of* A *is to the logarithm of* B. *And hence we have a way of meaſuring all ratios whatever, let their conſequents be what they will:* as for example, the ratio of A to B is the exceſs of the ratio of A to 1 above the ratio of B to 1, (ſee art. 296;) therefore the numeral exponent of the ratio of A to B will be the exceſs of the numeral exponent of the

ratio of A to 1 above the numeral exponent of the ratio of B to 1, that is, the excess of the logarithm of A above the logarithm of B; therefore *The magnitude of the ratio of* A *to* B *is to the magnitude of the ratio of* C *to* D *as the excess of the logarithm of* A *above the logarithm of* B, *which is the measure of the former ratio, is to the excess of the logarithm of* C *above the logarithm of* D, *which is the measure of the latter ratio:* and thus we see that logarithms are as true and as proper measures of ratios as circular arcs are of angles.

I might have defined logarithms from the idea here given of them, and thence have deduced all the other properties above described: but as it is not every one that hath a just and distinct notion of the nature and composition of ratios, I thought it more adviseable to treat of them in a way more familiar to the learner.

Of Briggs's *Logarithms.*

392. *From the definition given in art.* 390 *it may easily be seen, that if any one system of logarithms be once obtained, an infinite number of others may be derived from them by increasing or diminishing the logarithms of that system in some given proportion.* As for instance, in the system given let a, b, c be the logarithms of three numbers A, B and C, whereof the third is the product of the other two multiplied together; then will $a+b=c$, by the definition. Let us now imagine all the logarithms of this given system to be doubled; then will a, b and c be changed into $2a$, $2b$ and $2c$; but as $a+b$ was equal to c in the former system, so now will $2a+2b$ be equal to $2c$ in the latter; that is, all the numbers of this new system will still retain the property of logarithms. *But though all these different systems be equally perfect, if computed to the same degree of accuracy, yet they will not all be equally convenient for use; for of all systems or tables of logarithms, that is certainly best accomodated for practice which is now in use, and is commonly known by the name of* Briggs's *logarithms.* The Lord *Napeir*, a Scotch Nobleman, was the first inventer of logarithms; but our Countreyman Mr. *Briggs*, Professor of Geometry in *Gresham* College, was undoubtedly the first who thought of this system, and proposing it to the noble inventer, the Lord *Napeir*, he afterwards published it with that Lord's consent and approbation.

The distinguishing mark of this system is, that herein the logarithm of 10 *is* 1, *and consequently that of* 100, 2, *that of* 1000, 3, *that of* 10000, 4, &c; *that of* 1, 0, *that of* $\frac{1}{10}$ *or of* 0.1, -1, *that of* $\frac{1}{100}$ *or of* 0.01, -2, &c. *In this system the integral parts of the logarithms are always distinguished from the rest, and called the indexes or characteristics of the logarithms*

garithms whereof they are parts: thus the logarithm of 20 is 1.3010300, where the characteristic is 1; that of 2 is 0.3010300, where the characteristic is 0; that of $\frac{2}{10}$ or 0.2 is $-1+.3010300$, where -1 is the characteristic, &c.

Some advantages of this system.

393. Some of the chief advantages of this system beyond all others, will appear from the following considerations.

1st. Whereas we have frequent occasion to multiply and divide by 10, 100, 1000, &c, this in this system is very readily performed, only by adding to or subtracting from the characteristic the numbers 1, 2, 3, &c; and as these are whole numbers, they can only influence the index or characteristic of a logarithm, without affecting the decimal part.

2dly. So long as the digits that compose any number are the same, and in the same order, whatever be their places with respect to the place of units, the decimal parts of the logarithm of such a number will always be the same. As for instance, let $4+l$ be the logarithm of this number 34567.89, where 4 is the characteristic, and l represents the sum of all the decimal parts; then will $5+l$ be the logarithm of 345678.9, $6+l$ that of 3456789, $7+l$ that of 34567890, &c. On the other hand, $3+l$ will be the logarithm of 3456.789, $2+l$ that of 345.6789, $1+l$ that of 34.56789, $0+l$ that of 3.456789, $-1+l$ that of 0.3456789, $-2+l$ that of 0.03456789, &c: the reason of this is plain; for if the number 34567.89 be multiplied by 10, the product will be 345678.9; therefore if to $4+l$, the logarithm of the former number, be added 1, the logarithm of 10, the sum $5+l$ will be the logarithm of the latter. Again, if the number 34567.89 be divided by 10, the quotient will be 3456.789; therefore if from $4+l$, the logarithm of the former number, be subtracted 1, the logarithm of 10, the remainder $3+l$ will be the logarithm of the latter. Here then we see the reason why in *Briggs*'s tables, the decimal part of every logarithm is affirmative, whether the whole logarithm taken together be so or not; for in the logarithm of all numbers greater than unity, both the integral and decimal parts are affirmative; and therefore the decimal parts must always be so, since these are not changed by changing the natural number, so long as the digits that compose it are the same, and in the same order: thus $\frac{-3}{10}$ or $-.3$ may be a logarithm; but it is never expressed so, but rather thus, $-1+.7$, the negation being thrown wholly upon the characteristic.

3dly. By this means in *Briggs's* system the characteristic of the logarithm of any number is easily known thus: suppose I was asked, what is the characteristic of the logarithm of this number 34567.89? Here I consider that this number lies between 10000 and 100000; therefore it's logarithm must be some number between 4 and 5; therefore it must be 4 with some decimal parts annexed, that is, the characteristic must be 4. And again, suppose it was required to assign the characteristic of the logarithm of this number, 0.03456789: here I consider that this number lies between $\frac{1}{10}$ and $\frac{1}{100}$, that is, between 0.1 and 0.01, and therefore it's logarithm must lie between —1 and —2, that is, it's logarithm must be —2 with some affirmative decimal parts annexed, to lessen the negation; therefore the characteristic will be —2.

To find the characteristic of Briggs's *logarithm of any number.*

394. Hence may be drawn a short and easy rule for determining the index or characteristic of the logarithm of any number given, thus. *If the number given be a whole number, or a mixt number consisting of integral and decimal parts, then so many removes as is the place of units to the right hand of the first figure, of so many units will the characteristic consist: but if the number proposed be a pure decimal, then so many removes as is the place of units to the left hand of the first significant figure, of so many negative units will the characteristic consist.* Thus the index or characteristic of the logarithm of this number 34567.89 is 4, because 7 in the place of units is four removes to the right hand of the first figure 3: thus again, the characteristic of the logarithm of this number 0.03456789 is —2, because 0 in the place of units is two removes to the left hand of the first significant figure 3.

These rules are the more to be observed, because in some tables the integral parts of all logarithms are omitted, being left to be supplied by the operator himself, as occasion requires: by this means, the logarithms become of much more general use than if, by having their characteristics prefixed, they were tied down to particular numbers.

Another idea of logarithms.

395. *In the system here described, every natural number is, or may be considered as some power of* 10, *and it's logarithm as the index of that power:* for let *a* be the logarithm of any natural number as *A*; then since *Briggs's* logarithm of 10 is 1, his logarithm of 10^a will be *a*; this evident from

art.

art. 390 confect. 6; therefore A muft be equal to 10^a, fince they have both the fame logarithm; that is, the natural number A is fuch a power of 10 as is expreffed by it's logarithm a. This confideration gives us a new idea of logarithms, and to one acquainted with the nature of powers and their indexes, it will be no wonder that the addition, fubtraction, multiplication and divifion of thefe logarithms anfwer to the multiplication, divifion, involution and evolution of their natural numbers.

Precautions to be ufed in working by Briggs's *logarithms.*

396. Though thefe logarithms (as I obferved before) are preferable to all others, on account of their fimplicity and facility in practice, yet in ufing them fome precautions are to be obferved, which (to prevent miftakes) I fhall here juft point out to the learner; as

1ft, *In the addition of logarithms, whatever is carried over from the decimal to the integral parts, muft be confidered as affirmative, and as fuch muft be added to thofe integral parts, whether they be affirmative or negative.* Thus $-3+.7000000$ being added to $-4+.8000000$, the fum will be $-6+.5000000$; for though the fum of the characteriftics -3 and -4 be -7, the affirmative unit drawn from the decimals reduces it to -6.

2dly, *Whenever a fubtraction is to be made in logarithms, it muft be performed in the decimal parts as ufual; but if the characteriftic of the fubtrahend, or of the number from whence the fubtraction is to be made, or of both be negative, they muft be treated in the fubtraction as the nature of fuch quantities requires.* Thus $-3+.8900000$ fubtracted from $-1+.7600000$ leaves 1.8700000: for if $+1$, on account of the decimals, be added to -3, the characteriftic of the fubtrahend, it will be reduced to -2, which being fubtracted from -1 as above, leaves $+1$. Nay the learner muft not be difcouraged if he fometimes finds himfelf obliged to fubtract a greater logarithm from a lefs, as will always be the cafe where the logarithm of a proper fraction is required: as for example, let it be required to find the logarithm of $\frac{1}{2}$: here fubtracting 0.3010300, the logarithm of 2, from 0.0000000, the logarithm of 1, there will remain $-1+.6989700$, the logarithm of $\frac{1}{2}$; for in this fubtraction, $+1$ on account of the decimals being added to the characteriftic of the fubtrahend, gives 1, which fubtracted from 0 above, leaves -1.

Note, *The logarithm of a vulgar fraction may alfo be obtained by throwing it into a decimal.* Thus the logarithm of $\frac{2}{3}$ may be obtained, either by fubtracting the logarithm of 3 from that of 2, or elfe by taking out the logarithm of this decimal fraction .6666667, which is the fame as the logarithm of the whole number 6666667, except that the characteriftic of the former logarithm is -1, and that of the latter $+6$.

3dly,

3dly, *In the multiplication of logarithms the same care must be taken as in addition.* Thus if it be required to multiply this logarithm —3+.7000000 by 9, the product will be —21+.3000000; for though the product of —3 × 9 be —27, yet the +6 drawn from the decimals reduces it to —21.

4thly, *Whenever a logarithm is to be divided by* 2, 3, 4, &c *in order to obtain the square, cube, biquadrate* &c *root of it's natural number, if the characteristic be negative, and will not be divided without a fraction, my way is to resolve it into two parts, to wit, into a negative part which will be divided, and an affirmative part which will incorporate with the decimals annexed.* Thus if I was to take the half of this logarithm —1+.7000000, I cannot join the —1 to the decimals annexed, because they are quantities of different kinds; therefore I resolve the characteristic —1 into two parts, to wit, —2+1, and then taking the half of —2, which is —1, I join the affirmative part +1 to the decimals annexed, and so take the half of +17, which is +8 *&c*; therefore the half of the aforesaid logarithm is —1+.8500000: had the characteristic been —3, I should have resolved it into —4+1. Had ⅓ of the foresaid logarithm been required, I should have resolved the characteristic —1 into —3+2, and so should have taken, first, the third part of —3, which is —1, and then of +27, which is +9: had the characteristic been —2, I should have resolved it into —3+1; had it been —4, I should have resolved it into —6+2, and so on.

N. B. Of all the tables hitherto in use whose logarithms do not run to above seven decimal places, I take those published by Doctor *Sherwin* to be the best upon many accounts, and particularly in the disposition of the logarithms: these therefore I shall not scruple to recommend to my readers, whom I shall also refer to the directions there given for finding the logarithms of all absolute numbers from 1 to 10000000, and *vice versa.* But I must own I cannot with equal justice recommend the method there taken to avoid negative indexes by creating of new ones, and by using arithmetical complements. It is not to be denied but that this sort of practice may be absolutely necessary to such as know nothing of the nature and use of negative quantities; but those who do, I believe, will find the rules here laid down more natural and convenient; and as they carry their own reasons along with them, I doubt not but that the learner will find them easier to be remembered, and less liable to be misunderstood.

397. In the tables above recommended, after the logarithms on every page, are two columns, one called a column of differences, and signed *D*, the other called a column of proportional parts, and signed *Pts* above, and

and *Pro* below: these two columns, as well as the rest, have been explained by the author; but lest they should not be thoroughly understood by what is there said of them, I shall take the liberty by a single instance, to explain more at large the reason and use of these columns: I shall take my example from the author himself. Let it then be required to find by the tables the logarithm of this number of seven places, to wit, 5423758: to do this, I first put down 6, the characteristic of the logarithm sought, according to the directions given in art. 394; then I consider in the next place, that though by the help of the tables we can find the logarithm of any number under 10000000, yet that the absolute numbers there do not, properly speaking, run to above five places; therefore I lower the absolute number given, to wit, 5423758, to this 54237.58, which will not affect the decimal part of the logarithm sought; then setting aside the characteristic, I take out of the tables the logarithm of the five integral places 54237 according to the directions there given, and find it to be 7342957; this I subtract from the logarithm of 54238, that is, from 7343037, and find the difference to be 80. But the design of the column of differences is on purpose to avoid this subtraction; for had I taken out of that column the number opposite to 54237, the integral part of the absolute number proposed, or if no such opposite number was to be found, had I taken the nearest number above, (not below,) I should have found the number 80.1, that is, in a whole number, 80, without any subtraction. Thus then the case stands: as the absolute number proposed 54237.58 lies between the two nearest tabular numbers 54237 and 54238, whose difference is 1, so must the logarithm sought lie between the logarithms of the tabular numbers above mentioned, whose difference is 80; therefore I say by the golden rule, as 1, the difference of the two tabular numbers between which mine lies, is to 80, the difference of the two tabular logarithms between which the logarithm sought lies, so is .58, the difference betwixt my number and the nearest less tabular number, to 46, the difference betwixt the logarithm sought and the nearest less tabular logarithm; therefore adding this difference 46 to the nearest less tabular logarithm, to wit, 7342957, I have 7343003, which being joined as decimal parts to the characteristic 6, gives 6.7343003 for the logarithm sought. This number 46, which was the fourth proportional above found, is called the proportional part, because it is the same proportional part of 80, the difference of the two nearest tabular logarithms, that .58, the decimal part of the number proposed, is of 1, the difference of the two nearest tabular numbers. Whoever attends to the foregoing operation will easily perceive, that this proportional part 46 was gained from multiplying 80, the common difference, by .58, the deci-

mal parts of the abſolute number propoſed; and the ſame would have been obtained if the common difference 80 had firſt been multiplied by .5 and then by .08, and the products been taken into one ſum: now it is to ſave theſe two multiplications that the column of proportional parts was contrived; for whoever looks there for the common difference 80 will find all the products of the ſaid common difference multiplied by .1, .2, .3, .4, .5 &c to .9 incluſively; and looking for the number over againſt .5, he will find the number 40, which ſhews that the number 40 is $\frac{5}{10}$ of the common difference 80; ſo alſo over againſt 8 he will find the number 64, which ſhews that the number 64 is $\frac{8}{10}$ of the common difference; but we do not want $\frac{8}{10}$ of it, but 8 hundredth parts; therefore he muſt not take the number 64, but a tenth part of that number, to wit, 6.4 or 6, which being added to 40, the proportional part before found, gives 46, to be added to the neareſt leſs tabular logarithm in order to obtain the logarithm ſought.

But when all poſſible exactneſs is required, and no errors are intended to be committed, but ſuch as unavoidably ariſe from the imperfection of the logarithms themſelves, I would adviſe the reader to compute the proportional parts himſelf, as above, rather than truſt to the table for them, though he will rarely find any conſiderable difference. My reaſon for this advice is, becauſe in the table of proportional parts, no notice is taken of decimals, whereas thoſe decimals ought not in all caſes to be neglected, at leaſt not till the operation is over, and the artiſt ſees what it is he throws away or takes into his account, to leſſen the error as much as he can.

Thus having finiſhed all I thought proper to premiſe concerning the nature of logarithms in general, and of *Briggs*'s ſyſtem in particular, I ſhall now proceed to logarithmotechny, or the art of computing theſe logarithms; and herein I ſhall have little regard to the methods made uſe of by the firſt inventers, which may be ſeen in *Briggs* and others, as being infinitely more laborious and perplexed than the modern ways of effecting the ſame. The method I ſhall proceed in is the eaſieſt I could think of, both for myſelf and my readers; for I ſhall endeavour to make the whole ſubſiſt upon the feweſt principles poſſible, and thoſe either ſuch as have been already explained, or will be explained in the courſe of the work; and in doing of this I ſhall endeavour to deliver my thoughts with all the plainneſs and perſpicuity the nature of ſuch a ſubject will admit.

Of

Of logarithmotechny, or the construction of logarithms.

PROPOSITION I.

398. *In the same system, the evanescent logarithms of all numbers gradually approaching towards unity are as the differences of those numbers from unity.*

Let there be two quantities $1+z$ and $1+y$, whose differences from unity, to wit z and y, are supposed to lessen by degrees, and at last to vanish in some finite ratio: I say then that the logarithms of $1+z$ and $1+y$ will vanish with them, and in the same ratio.

That the logarithms of $1+z$ and $1+y$ will vanish at the same time with the quantities z and y is evident from hence, that when z and y are actually vanished, the quantities $1+z$ and $1+y$ will become each equal to unity; and therefore the logarithm of each will be nothing, by art. 390, consect. 1: therefore what I am chiefly to demonstrate in this proposition is, that the ultimate ratio of the evanescent logarithms of $1+z$ and $1+y$ will be the same with the ultimate ratio of the evanescent quantities z and y.

But before I enter upon this demonstration, I shall beg leave to remind the reader of one thing which I have in some measure taken notice of in another place, to wit, that though two quantities, after they have lost their existence, be equally nothing, or equally in a state of nothingness, yet it does not necessarily follow from hence that they must vanish in a ratio of equality.

Let a parallelogram and a triangle, having the same base and being terminated by the same parallels, be made to vanish by lessening either their common base, or their common altitude, or both; then will these two quantities after they are vanished be equally in a state of nothingness; but if in every instant of their existence the parallelogram was double of the triangle, it must be so in the very last of all, and so these quantities will vanish in the ratio of 2 to 1. For a fuller account of evanescent ratios see art. 336.

N. B. Time is made up of moments as a line is made up of lesser lines; but an instant is to time what a mathematical point is to a line; it is called in Latin *articulus temporis*, as being itself no part of time, but the meeting of two parts of time immediately succeeding one another, and thereby distinguished one from the other: and it is in this sense Mathematicians are to be understood, when they suppose a quantity to be *changed* in a moment, but to *vanish* in an instant. I shall now return to the demonstration of the proposition, which is as follows.

Let the ultimate ratio of z to y be that of 1 to m, and let the quantity $\overline{1+z}^m$ be resolved into a series by *Newton*'s theorem for the evolution of the powers of a binomial already explained, and you will have $\overline{1+z}^m = 1 + \frac{m}{1}z + \frac{m}{1} \times \frac{m-1}{2}z^2 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}z^3$ &c: put q for the sum of all the terms of this series except the first, that is, let $q = \frac{m}{1}z + \frac{m}{1} \times \frac{m-1}{2}z^2 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}z^3$ &c, and you will have $\overline{1+z}^m = 1+q$, and z will be to q as z is to $\frac{m}{1}z + \frac{m}{1} \times \frac{m-1}{2}z^2 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}z^3$ &c, or as 1 is to $\frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2}z$ &c. This is universal; but if we suppose the quantity z to vanish, then the term $\frac{m}{1} \times \frac{m-1}{2}z$ and all that follow it will vanish at the same time, and the ultimate ratio of z to q will be that of 1 to m: but the ultimate ratio of z to y is also that of 1 to m by the supposition; therefore in the last instant of their existence, y will be equal to q, in which case we shall have $1+y = 1+q = \overline{1+z}^m$, and the logarithm of $1+y$ equal to the logarithm of $\overline{1+z}^m$; but the logarithm of $1+z$ is to the logarithm of $\overline{1+z}^m$ as 1 to m, by art. 390, consect. 6; therefore the logarithm of $1+z$ is to the logarithm of $1+y$ as 1 to m; that is, the ultimate ratio of the evanescent logarithms of $1+z$ and $1+y$ is the same with the ultimate ratio of the evanescent quantities z and y.

In like manner it is demonstrated that the ultimate ratio of the evanescent logarithms of $1-z$ and $1-y$ is the same with the ultimate ratio of the evanescent quantities $-z$ and $-y$; to wit, by throwing into a series the power m of the residual $1-z$, as before we did the same power of the binomial $1+z$.

This proposition might also have been demonstrated another way, founded upon the 298th article, thus: since the quantities $1+z$ and $1+y$ approach infinitely near to unity, it follows from what was observed towards the latter end of the article above quoted, that the difference z will be to the difference y as the quantity of the ratio of $1+z$ to 1 is to the quantity of the ratio of $1+y$ to 1: but the quantity of the ratio of $1+z$ to 1 is to the quantity of the ratio of $1+y$ to 1 as the measure of the former ratio is to the measure of the latter, that is, by art. 391, as the logarithm of $1+z$ is to the logarithm of $1+y$; therefore the logarithm of $1+z$ is to the logarithm of $1+y$ as z is to y. *Q. E. D.*

Since

Since ultimately the logarithm of $1+z$ *is to the logarithm of* $1+y$ *as* z *is to* y, *it follows alternately that the logarithm of* $1+z$ *is to* z *as the logarithm of* $1+y$ *is to* y. Here then we have a foundation for raiſing as many different ſyſtems of logarithms as we pleaſe, by aſſigning what ratio we pleaſe for the ultimate ratio of the evaneſcent logarithm of $1+z$ to the evaneſcent difference z. As for example, if in the laſt inſtant of their exiſtence we ſuppoſe the logarithm of $1+z$ to be equal to z, the logarithms derived from this ſuppoſition (as in the next article) are commonly called *Napeir*'s logarithms, by ſome natural logarithms, by others hyperbolic logarithms, from the relation they have to a certain property of the common *hyperbola*: but if we aſſume any quantity as p different from unity, and ſuppoſe the ratio of the logarithm of $1+z$ to the difference z to terminate at laſt in the ratio of p to 1, we ſhall then have a foundation for a ſyſtem of logarithms different from the former; for the logarithms of this ſyſtem will be to *Napeir*'s logarithms of the the ſame natural numbers as p to 1. Thus if p be taken equal to .43429448190З, we ſhall have a foundation for raiſing *Briggs*'s logarithms, as will be ſhewn in it's proper place.

PROPOSITION 2.

399. *To find* Napeir's *logarithm of any whole number or fraction whatever.*

CASE I.

Let z be any number leſs than unity, and let it be required to find *Napeir*'s logarithm of the number $1+z$.

SOLUTION.

Aſſuming any number as m for the index of a power, let the quantity $\overline{1+z}^m$ be expanded into a ſeries, *Newton*'s way, and you will have $1+\frac{m}{1}z+\frac{m}{1}\times\frac{m-1}{2}z^2+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}z^3$ &c: make $1+q$ equal to $\overline{1+z}^m$, and you will have $q=\frac{m}{1}z+\frac{m}{1}\times\frac{m-1}{2}z^2+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}z^3$ &c; divide both ſides by m, and you will have $\frac{q}{m}=z+\frac{m-1}{2}z^2+\frac{m-1}{2}\times\frac{m-2}{3}z^3$ &c: this equation is univerſal, be the index m what it will. Let us now ſuppoſe the index m to vaniſh

into

into nothing, and then the quantities $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$ &c will become $\frac{-1}{2}$, $\frac{-2}{3}$, $\frac{-3}{4}$ &c: for the index m being now infinitely ſmall in compariſon of the fractions joined with it, or rather nothing, it may be dropped on that ſide; but it muſt by no means be dropped on the other ſide, becauſe though an infinitely ſmall quantity has no effect when added to or ſubtracted from a finite one, yet it has an infinite effect when any quantity is multiplied or divided by it: therefore in the laſt inſtant of m we have $\frac{q}{m} = z + \frac{-1}{2}z^2 + \frac{-1}{2} \times \frac{-2}{3}z^3 +$ [illegible] $\times$ [illegible] $+ \cdot \frac{-1}{2} \times \frac{-2}{3} \times$ [illegible] $z^5 + \frac{-1}{2} \times \frac{-2}{3} \times \frac{-3}{4} \times \frac{-4}{5} \times \frac{-5}{6}z^6$ &c, where the ſign $+$ is only uſed to ſignify that the ſubſequent terms, whether affirmative or negative, are as ſuch to be added to the antecedent ones. But $\frac{-1}{2} \times$ [illegible] $+$ [illegible] therefore [illegible] or $+\frac{1}{}\times$ [illegible] $\frac{-3}{4}$ [illegible] therefore [illegible] $\times$ [illegible] $\times$ [illegible] $\frac{-4}{5}$ or $\frac{-4}{5}$ [illegible] $+1$ [illegible] therefore [illegible] $\times$ [illegible] $\frac{-3}{4} \times \frac{-4}{5} \times \frac{-5}{6}$ or $+\frac{1}{5} \times \frac{-5}{6} = \frac{-1}{6}$ &c: therefore in the laſt inſtant of m, the fraction $\frac{q}{m}$ is found to be the ſum of the following ſeries, $z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4 + \frac{1}{5}z^5 - \frac{1}{6}z^6 +$ &c; but the ſum of this ſeries is a finite quantity, which I thus demonſtrate. The ſum of the following ſeries, to wit, $z + z^2 + z^3 + z^4 + z^5 + z^6$ &c is a finite quantity, and equal to the fraction $\frac{z}{1-z}$, as will appear by dividing the numerator of that fraction by the denominator; but this laſt ſeries is greater than the former, and therefore the ſum of the former ſeries muſt either be a finite, or an infinitely ſmall quantity; but an infinitely ſmall quantity it cannot be; for the firſt couple of terms conſtitutes a finite quantity, and every ſucceeding couple makes ſome addition to the former; therefore when m is evaneſcent, the fraction $\frac{q}{m}$, or the ſum of the ſeries $z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \frac{z^6}{6}$ &c is a finite quantity; therefore the ultimate ratio of q to m is a finite ratio; whence it follows that when the index m vaniſhes, the quantity q will vaniſh with it: but q in it's evaneſcent ſtate is *Napeir*'s logarithm of $1+q$ by the laſt article, and conſequently of $\overline{1+z}$ by the ſuppoſition: but

but if *Napeir*'s logarithm of $\overline{1+z}^m$ be equal to q in it's laſt inſtant, then *Napeir*'s logarithm of $1+z$ will be $\frac{q}{m}$ by art. 390, conſect. 6, or the ſum of the ſeries $z-\frac{z^2}{2}+\frac{z^3}{3}\quad\frac{z^4}{4}+\frac{z^5}{5}\quad\frac{z^6}{6}+$ &c.

CASE 2.

Let us now change the ſign of z, and by this means the ſigns of all the odd powers of z will be changed; but this will not affect the even powers, and therefore they will continue the ſame as before, and we ſhall have *Napeir*'s logarithm of $1-z$ expreſſed by the following ſeries,

$z^2 \quad z^3 \quad \frac{z^4}{4}$ &c.

CASE

Subtract the logarithm of $1-z$ in the ſecond caſe from the logarithm of $1+z$ in the firſt, and there will remain *Napeir*'s logarithm of the fraction $\frac{1+z}{1-z}$ by art. 390, conſect. 4, to wit, $2z+\frac{2z^3}{3}+\frac{2z^5}{5}+\frac{2z^7}{7}+\frac{2x^9}{9}+$ &c, which ſeries converges much faſter than either of the former. Now ſince $2z\times z^2=2z^3$, and $2z^3\times z^2$ gives $2z^5$ &c, we ſhall have the following theorem for computing *Napeir*'s logarithm of the fraction $\frac{1+z}{1-z}$. *Make* $2z=A$, $Az^2=B$, $Bz^2=C$, $Cz^2=D$, $Dz^2=E$ &c, *and* Napeir's *logarithm of the fraction* $\frac{1+z}{1-z}$ *will be* $A+\frac{B}{3}+\frac{C}{5}+\frac{D}{7}+\frac{E}{9}$ &c.

CASE 4.

Let now any fraction as $\frac{r}{s}$ be propoſed, whoſe numerator is greater than the denominator, and let it be required to find *Napeir*'s logarithm of this fraction. Make $\frac{r}{s}=\frac{1+z}{1-z}$, and reſolving the equation, you will have $z=\frac{r-s}{r+s}$: *Make therefore* $\frac{r-s}{r+s}=z$, $2z=A$, $Az^2=B$, $Bz^2=C$,

Cz^2

$Cz^2=D$, $Dz^2=E$ &c, *and you will have* Napeir's *logarithm of* $\frac{r}{s}$ *or of* $\frac{1+z}{1-z}=A+\frac{B}{3}+\frac{C}{5}+\frac{D}{7}+\frac{E}{9}$ &c.

N. B. The less the quantity $\frac{r-s}{r+s}$ or z is in respect of unity, the faster the series *A, B, C, D, E &c* will converge.

CASE 5.

Let now the fraction $\frac{s}{r}$ be proposed, whose numerator is less than the denominator. Now to find *Napeir*'s logarithm of this fraction, first invert the terms, and then find the logarithm of $\frac{r}{s}$ by the last case; this done, prefix the negative sign to the logarithm thus found, and you will have the logarithm of the fraction $\frac{s}{r}$: the reason is, because these two fractions $\frac{r}{s}$ and $\frac{s}{r}$ being multiplied together produce $\frac{rs}{rs}$ or 1; therefore their logarithms added together make nothing; therefore the logarithm of $\frac{s}{r}$ is the negative logarithm of $\frac{r}{s}$.

PROPOSITION 3.

400. *To compute* Napeir's *logarithm of* 10 *by the help of the foregoing proposition.*

Briggs's logarithm of 10 is known already, being intended to be 1, and consequently his logarithm of 100, 1000, 10000 *&c* is 2, 3, 4 *&c* respectively: but before any of the rest can be obtained, (for at present we are to suppose no such logarithms as yet in being,) it will be necessary to compute *Napeir*'s logarithm of the same number; that so by observing the proportion of these two logarithms, and consequently of all others of the same natural numbers, we may be the better able to furnish out a series for the computation of *Briggs*'s logarithms. Now if *Napeir*'s logarithm of 10 or $\frac{10}{1}$ was to be computed at one operation by the help of the theorem laid down in the fourth case of the foregoing proposition, the work would be found intolerably tedious; because the number

number z, which in this case is $\frac{10-1}{10+1}$ or $\frac{9}{11}$ is so near unity, that the series would converge too slow: let us therefore try whether we cannot succeed better at two operations, thus. The number 10 may be considered as the product of 8 multiplied into $\frac{10}{8}$ or $\frac{5}{4}$; therefore the logarithm of 10 is made up of the logarithm of 8 and the logarithm of $\frac{5}{4}$ put together; but the logarithm of 8 is three times the logarithm of 2, because 8 is the third power of 2: therefore the whole business is now reduced to the computation of these two logarithms, to wit, the logarithm of 2 and the logarithm of $\frac{5}{4}$, the former of which may be had without much trouble, and the latter with a great deal of ease.

First for the logarithm of 2 or $\frac{2}{1}$. Here it is plain that z or $\frac{2-1}{2+1}$ $=\frac{1}{3}$, and $z^2=\frac{1}{9}$; therefore if we make $2z$ or $\frac{2}{3}=A$, Az^2 or $\frac{A}{9}=B$, $\frac{B}{9}=C$, $\frac{C}{9}=D$ &c, we shall have *Napeir's* logarithm of 2 equal to $A+\frac{B}{3}+\frac{C}{5}+\frac{D}{7}+\frac{E}{9}+\frac{F}{11}+\frac{G}{13}+\frac{H}{15}$ &c: or if we make $2=A$, $\frac{A}{9}=B$, $\frac{B}{9}=C$, $\frac{C}{9}=D$ &c, we shall have *Napeir's* logarithm of 8 equal to $A+\frac{B}{3}+\frac{C}{5}+\frac{D}{7}+\frac{E}{9}+\frac{F}{11}+\frac{G}{13}+\frac{H}{15}$ &c; for by making $A=2$ instead of $\frac{2}{3}$, the first term of the series, and consequently all the rest, are multiplied by 3; see art. 383.

Next for calculating the logarithm of $\frac{5}{4}$, we have z or $\frac{5-4}{5+4}=\frac{1}{9}$, and $z^2=\frac{1}{81}$; therefore if we make $\frac{2}{9}=K$, $\frac{K}{81}=L$, $\frac{L}{81}=M$, $\frac{M}{81}$ $=N$ &c, we shall have *Napeir's* logarithm of $\frac{5}{4}=K+\frac{L}{3}+\frac{M}{5}+\frac{N}{7}$ &c. But here it luckily happens that every term of this series is equal to every other term of the foregoing series expressing the logarithm of 8: for beginning with the term B, we shall have $K=B$, $L=D$, $M=F$, $N=H$ &c, which I thus demonstrate. $K=\frac{2}{9}=\frac{A}{9}=B$; $L=\frac{K}{81}$ $=\frac{B}{81}=\frac{C}{9}=D$; $M=\frac{L}{81}=\frac{D}{81}=\frac{E}{9}=F$; $N=\frac{M}{81}=\frac{F}{81}=\frac{G}{9}=H$, &c. Therefore in calculating the logarithm of $\frac{5}{4}$ we shall have no occasion for any new series, but may take every other term of the series al-

 ready

ready computed for the logarithm of 8, beginning with the term B; and fo we fhall have *Napeir*'s logarithm of $\frac{5}{4}$ equal to $B+\frac{D}{3}+\frac{F}{5}+\frac{H}{7}$ &c.

N. B. If we would have a logarithm true to any number of places, it will be proper to compute it to a place or two more than are intended to be true, that an error in the laft place, or perhaps in the laft but one, may not influence the reft.

See the work, where B is the quotient of A divided by 9, C that of B divided by 9, and fo on.

A	2.00000000000000.
B	22222222222222.
C	2469135802469.
D	274348422497.
E	30483158055.
F	3387017562.
G	376335285.
H	41815032.
I	4646115.
K	516235.
L	57359.
M	6373.
N	708.
O	79.
P	9.
Q	1.

A	2.00000000000000.
$\frac{B}{3}$	7407407407407
$\frac{C}{5}$	493827160494.
$\frac{D}{7}$	39192631785
$\frac{E}{9}$	3387017562
$\frac{F}{11}$	307910687
$\frac{G}{13}$	28948868
$\frac{H}{15}$	2787669.
$\frac{I}{17}$	273301.
$\frac{K}{19}$	27170.
$\frac{L}{21}$	2731.
$\frac{M}{23}$	277.
$\frac{N}{25}$	28.
$\frac{O}{27}$	3.
Log. 8.	207944154167982

B	22.2222222222222.
$\frac{D}{3}$	91449474166.
$\frac{F}{5}$	677403512.
$\frac{H}{7}$	5973576.
$\frac{K}{9}$	57359.
$\frac{M}{11}$	579.
$\frac{O}{13}$	6.
Log. 10.	2.30258509299402.

N. B. Napeir's true logarithm of 10 lies between 2.302585092994 04 and 2.302585092994 05.

PRO-

PROPOSITION 4.

401. *To find* Briggs's *logarithm of any whole number or fraction whatever.*

SOLUTION.

Briggs's logarithm of any number is to *Napeir*'s logarithm of the same number as *Briggs*'s logarithm of 10 is to *Napeir*'s logarithm of 10; that is, by the last proposition, as 1 is to 2.302585093, or as $\frac{1}{2.302585093}$ is to 1, that is, throwing the fraction into decimals, as .4342944819 is to 1. Make this number $.4342944819 = p$, and then we shall have *Briggs*'s logarithms to *Napeir*'s logarithms as p to 1; that is, if any of *Napeir*'s logarithms be multiplied by p, the product will be *Briggs*'s logarithm of the same number: but *Napeir*'s logarithms will be multiplied by p, if the first term of the series which produces them be so multiplied, by art. 383. *If therefore* $\frac{r}{s}$ *be the fraction proposed, and* z *be taken equal to* $\frac{r-s}{r+s}$, *instead of making* $2z = A$ *as before, make* $2pz = A$, $Az^2 = B$, $Bz^2 = C$ &c, *and you will have* Briggs's *logarithm of* $\frac{r}{s}$ *equal to* $A + \frac{B}{3} + \frac{C}{5}$ &c. As for example, let it be required to find *Briggs*'s logarithm of the number 2, or of the fraction $\frac{2}{1}$: here I might make $z = \frac{1}{3}$, $z^2 = \frac{1}{9}$, $2pz$ or $\frac{2p}{3} = A$, $\frac{A}{9} = B$, $\frac{B}{9} = C$ &c, as before; but for variety's sake I shall take another way for finding this logarithm, which is also more expeditious. First then I seek out for two powers of the numbers 10 and 2 which do not differ very much from one another; and such are the numbers 1000 and 1024, the former being the third power of 10, and the latter the tenth power of 2; then since the number 1024 is the product of 1000 multiplied into the fraction $\frac{1024}{1000}$, it follows that the logarithm of 1024 is made up of the logarithm of 1000 and the logarithm of $\frac{1024}{1000}$ added together: but *Briggs*'s logarithm of 1000 is known already to be 3; if therefore *Briggs*'s logarithm of $\frac{1024}{1000}$ be

be computed, and added to the number 3, you will have *Briggs*'s logarithm of 1024; the tenth part whereof will be the logarithm of 2, because, as was said before, the number 1024 is the tenth power of 2. Here then $\frac{r}{s}=\frac{1024}{1000}$, $\frac{r-s}{r+s}$ or $z=\frac{24}{2024}$ or $\frac{3}{253}$, $z^2=\frac{9}{64009}$, $2pz$ or $\frac{6p}{253}=A$, $\frac{9A}{64009}=B$, $\frac{9B}{64009}=C$ &c. But $p=.4342944819$; therefore $\frac{6p}{253}$ or $A=.0102994739$, $\frac{9A}{64009}$ or $B=.0000014482$, $\frac{9B}{64009}$ or $C=.0000000002$; $\frac{B}{3}=.0000004827$, $\frac{C}{5}=.0000000000$; therefore $A+\frac{B}{3}$, or the logarithm of $\frac{1024}{1000}$ is .0102999566; add to this the logarithm 1000, which is 3, and you will have the logarithm of 1024 equal to 3.0102999566, a tenth part whereof is *Briggs*'s logarithm of 2, to wit, .30102999566, all which eleven places are true; see *Briggs*'s logarithm of 2, folio edition. But if seven of the first places be taken, as in the common tables, and so taken as to make the least error, we shall then have *Briggs*'s logarithm of 2 equal to .3010300.

PROPOSITION 5.

402. *To construct a table of* Briggs's *logarithms of all the natural numbers from 1 to 100000.*

N. B. To shorten, or rather to free our expressions from the incumberance of words, we shall make use of the letter *L.* to signify *the logarithm of:* thus *L.* 2 signifies the logarithm of 2, *L.* $\frac{1}{2}$ signifies the logarithm of the fraction $\frac{1}{2}$, *L.* 8 = 3 *L.* 2 signifies that the logarithm of 8 is equal to three times the logarithm of 2, &c.

Whoever undertakes to construct a table of logarithms, must not think of doing it *per saltum*, but by a regular progress from one number to another. The logarithms of all composite numbers, that is, all such as arise from the multiplication of others called factors, are easily got by the addition of the logarithms of those factors: thus $L.4=2L.2$; $L.10=L.2+L.5$, and therefore $L.5=L.10-L.2$; $L.6=L.2+L.3$; $L.8=3L.2$; $L.9=2L.3$, &c; so that by the help of the logarithms of the three prime numbers 2, 3 and 7 may be had the logarithms of all numbers under 11: now of these three logarithms, that of 2 was calculated in the last proposition, and those of 3 and 7 may also be had directly from the same proposition, to wit, by considering the number 3 as the product of $2\times\frac{3}{2}$, and so resolving it's logarithm into $L.2+L.\frac{3}{2}$, and

and the number 7 as the product of $6\times\frac{7}{6}$, and so resolving it's logarithm into $L.6+L.\frac{7}{6}$. But because the series $A+\frac{B}{3}+\frac{C}{5}$ &c, exhibiting the logarithms of $\frac{1}{2}$ and $\frac{7}{6}$ &c, converges slowest at the beginning of the table of logarithms, I shall propose other expedients for finding these logarithms; and first for the logarithm of 3.

The number 81 is the product of $80\times\frac{81}{80}$; therefore $L.81=L.80+L.\frac{81}{80}$; but $L.80=L.8+L.10$ or $3L.2+L.10$; and $L.\frac{81}{80}$ may be very readily computed by the series of the last proposition; for if the logarithms be computed to ten or eleven places, in order to have seven of the first places true, which we shall all along suppose in these calculations, the two first terms of the series, to wit $A+\frac{B}{3}$, will be sufficient; for the next term $\frac{C}{5}$ (which is .00000000000 &c) can only affect the twelfth decimal place and those that follow it. Having now got the logarithms of 80 and $\frac{81}{80}$, their sum will be the logarithm of 81, and $\frac{1}{4}$ of that sum the logarithm of 3, because 81 is the fourth power of 3.

The logarithm of 7 may be had most compendiously thus: the fourth power of 7 is 2401; therefore $4L.7=L.2401$; therefore $L.7=\frac{L.2401}{4}$; but $2401=2400\times\frac{2401}{2400}$; therefore $L.2401=L.2400+L.\frac{2401}{2400}$: now 2400 is the product of $2\times2\times2\times3\times100$; therefore $L.2400=3L.2+L.3+2$; therefore $L.2400$ is known: it remains then that we compute $L.\frac{2401}{2400}$, by making $z=\frac{2401-2400}{2401+2400}=\frac{1}{4801}$; then we shall have $z^2=\frac{1}{23049601}$; therefore in this case, $2pz$ or $A=\frac{.868588963806}{4801}=.000180918342$8, Az^2 or $B=.0000000000078$; whence $\frac{B}{3}=.0000000000026$: whence it appears that the first term A will give $L.\frac{2401}{2400}$ true to eleven places, (and the first and second terms $A+\frac{B}{3}$ to several places more;) and this logarithm added to $L.2400$ will give $L.2401$, whose fourth part is $L.7$.

Before I proceed any further, give me leave to observe that the logarithm of 3 might have been computed independently of all other loga-

rithms

rithms but that of 10, thus: $9 \times \frac{10}{9} = 10$; therefore $L.9 + L.\frac{10}{9} = 1$; therefore $1 - L.\frac{10}{9} = L.9$, whose half is $L.3$: here then we must compute $L.\frac{10}{9}$, by making $z = \frac{1}{19}$ and $z^2 = \frac{1}{361}$; for then we shall have $2pz$ or $A = .045715208621$, Az^2 or $B = .000126634927$, Bz^2 or $C = .000000350789$, Cz^2 or $D = .000000000972$, Dz^2 or $E = .000000000003$: whence we have again

$$\begin{aligned} A &= .045715208621. \\ \frac{B}{3} &= .000042211642. \\ \frac{C}{5} &= .000000070158. \\ \frac{D}{7} &= .000000000139. \\ \hline L.\frac{10}{9} &= .045757490560. \end{aligned}$$

Subtract this last from $L.10$ or 1.000000000000, and you will have $L.9 = .954242509440$, and $L.3 = .477121254720$, all which twelve places are true.

The ten first logarithms being now computed, or being supposed to be computed, there will be no occasion after this for any other artifice than that which follows. Let n be any prime number, and let the logarithms of all numbers less than n be supposed to be known: now since n is a prime number, and consequently cannot be divided by 2, it must also be an odd number; therefore the next number above it, $n+1$, must be an even number; therefore $\frac{n+1}{2}$ must be a whole number; therefore $L.n+1$ must be looked upon as known, being equal to $L.2 + L.\frac{n+1}{2}$: but $L.n-1$ is known *ex hypothesi*; therefore the logarithm of $\overline{n-1} \times \overline{n+1}$, or of $nn-1$ will be given: but the logarithm of nn is equal to the logarithm of $nn-1$ added to the logarithm of $\frac{nn}{nn-1}$; compute therefore this last logarithm by the series, to wit, the logarithm of $\frac{nn}{nn-1}$, making $\frac{1}{2nn-1} = z$; and this added to the given logarithm of $nn-1$ will give the logarithm of nn, half whereof is the logarithm of n. *Ex. gr.* the next prime number whose logarithm is to be found is 11; now $L.10$ is known *ex hypothesi*, and $L.12 = L.2 + L.6$; therefore $L.10 + L.12$ or $L.120$ is given: but $11^2 = 121$; compute

compute therefore by the series $L.\frac{121}{120}$, making $\frac{1}{241}=z$, and this logarithm added to $L.$ 120 will give $L.$ 121, whose half is $L.$ 11; and in this calculation, and consequently in all those that follow, there will be no occasion for more than the two first terms of the series. And if this method be pursued, after forty of the first logarithms are obtained, the first term alone of the series will be sufficient; for the next prime number being 41, whose square is 1681, the next logarithm to be calculated will be that of $\frac{1681}{1680}$, where the first significant figure of the second term $\frac{B}{3}$ hath eleven decimal cyphers before it. And thus the operator will find his labour diminish so fast upon his hands, that 1200 logarithms being thus obtained, he may proceed to the rest in a more direct manner; that is, he may find the logarithm of 1201 only by calculating the logarithm of the fraction $\frac{1201}{1200}$, and then adding this logarithm to the logarithm of 1200, for which purpose the first term A of the series will be sufficient to give the logarithm of 1201 true to ten decimal places.

Thus may an intire system of logarithms from 1 to 100000 be obtained true to as many decimal places as we please; but this (thanks to our predecessors) is already done to our hands. Mr. *Briggs* has left us the logarithms of thirty one chiliads (from one to 20 thousand, and from 90 thousand to 101 thousand) to fourteen decimal places; and M. *Vlacq* has given us an intire canon of logarithms to ten decimal places; and certain it is that nothing but the extraordinary usefulness of these numbers could ever have obliged the undertakers to such indefatigable pains and industry as they employed in their way of obtaining them. And yet this doctrine of logarithms was not so thoroughly cultivated by the first inventers, but that the moderns have found many and great uses of these numbers that were not so much as thought of by them: in Algebra, as in the resolution of cubic, and consequently biquadratic equations; in determining the *unciæ* of the higher powers of a binomial; in the resolution of many curious problems relating to chances, &c: in Geometry, as in the quadrature of binomial and trinomial curves; in the rectification of curves; in the mensuration of surfaces, &c: in Mechanics, as in determining the centers of gravity of many bodies; in determining the nature and properties of several mechanical curves, as the *catenaria* and others: in natural Philosophy, as in determining the density of the atmosphere at all altitudes, and upon all *hypotheses* of gravity; in determining the resistance of fluids in all cases of motion, &c.

PRO-

PROPOSITION 6.

403. *To find the natural number to any of* Napeir's *logarithms given.*

SOLUTION.

Let l be the given logarithm, and let m be any indefinite number supposed to be increased *ad infinitum*; then will it's reciprocal $\frac{l}{m}$ be diminished *ad infinitum*, so as at last to become *Napeir*'s logarithm of the number $1+\frac{l}{m}$: but if $1+\frac{l}{m}$ be a number whose logarithm is $\frac{l}{m}$, then $\overline{1+\frac{l}{m}}^{m}$ will be the number whose logarithm is l; therefore the number sought is the last magnitude of the quantity $\overline{1+\frac{l}{m}}^{m}$ when the index m is increased *in infinitum*. Now the quantity $\overline{1+\frac{l}{m}}^{m}$ when expanded into a series according to *Newton*'s theorem for evolving a binomial already explained, will be $1+\frac{m}{1}A\frac{l}{m}+\frac{m-1}{2}B\frac{l}{m}+\frac{m-2}{3}C\frac{l}{m}+\frac{m-3}{4}D\frac{l}{m}$ &c; of which series the second term $\frac{m}{1}A\frac{l}{m}=A\frac{l}{1}$, the third term $\frac{m-1}{2}B\frac{l}{m}=\frac{m-1}{m}B\frac{l}{2}=\overline{1-\frac{1}{m}}\times B\frac{l}{2}$; in like manner, the fourth term $\frac{m-2}{3}C\frac{l}{m}=\overline{1-\frac{2}{m}}\times C\frac{l}{3}$ &c. Therefore $\overline{1+\frac{l}{m}}^{m}$ $=1.+A\frac{l}{1}.+\overline{1-\frac{1}{m}}\times B\frac{l}{2}.+\overline{1-\frac{2}{m}}\times C\frac{l}{3}.+\overline{1-\frac{3}{m}}\times D\frac{l}{4}$ &c. This is universal; but if we suppose the index m to become actually infinite, the quantities $\frac{1}{m}$, $\frac{2}{m}$, $\frac{3}{m}$ &c will become actually nothing, and *The last magnitude of the quantity* $\overline{1+\frac{1}{m}}^{m}$, *which is the number sought, will be* $1+A\frac{1}{1}+B\frac{1}{2}+C\frac{1}{3}+D\frac{1}{4}$ &c $=1+\frac{1}{1}+\frac{l^2}{1\times 2}+\frac{l^3}{1\times 2\times 3}+\frac{l^4}{1\times 2\times 3\times 4}$ &c. Q. E. I.

Whence

Whence e converso, *whenever we meet with a series of this kind, to wit,* $1 + \frac{l}{1} + \frac{l^2}{1\times 2} + \frac{l^3}{1\times 2\times 3} + \frac{l^4}{1\times 2\times 3\times 4}$ &c, *the sum of such a series will be the natural number to* Napeir's *logarithm* l, *or to* Briggs's *logarithm* l×.43429448190З. *But if the terms of the series be alternately affirmative and negative, as* $1 - \frac{l}{1} + \frac{l^2}{1\times 2} - \frac{l^3}{1\times 2\times 3} + \frac{l^4}{1\times 2\times 3\times 4} -$ &c, *then the sum of the series will be the natural number to* Napeir's *logarithm* —l.

What has been delivered in these few propositions may serve as a specimen, amongst an infinite number of others that might be given, to shew by how easy, how simple, and natural steps this doctrine of infinites leads us into the most abstruse, as well as the most sublime parts of mathematical knowledge: and it were to be wished that those who are so very clamorous against it, would advance any thing like this from their idea of infinity to recommend it, which hitherto has produced nothing but jargon and confusion, noise and nonsense.

PROPOSITION 7.

404. *To find the logarithm of any number not exceeding ten places by* Vlacq's canon magnus.

This is the only intire system we have, that I know of, where the logarithms are continued so far as to ten decimal places: those indeed published by Mr. *Briggs* run to fourteen; but the chiliads are not compleat.

When the logarithm of any number is required, it is usual to take the logarithm belonging to the five first places out of the tables, and then to supply the rest from a proportional part of the difference, which, where the logarithms run not to above eight or nine places, may do well enough; but when they run further, the proportional part may want correcting; and to find that correction is the design of this proposition. But first I must demonstrate the following lemma, to wit, *Supposing* $p =$ .4342944819, *as in the fourth proposition, let* s *represent any number not less than* 10000, *and* t *any number not greater than unity; I say then that the first term of the series exhibiting the logarithm of the fraction* $\frac{s+t}{s}$ *to wit* $\frac{2pt}{2s+t}$, *will give that logarithm true to above thirteen places, and consequently to a much greater degree of exactness than* Vlacq's canon *requires.*

To demonstrate this, let us put the case where the first term $\frac{2pt}{2s+t}$ differs most from the true logarithm of the fraction $\frac{s+t}{s}$; that is, let us suppose $s=10000$ and $t=1$; then we shall have $\frac{s+t}{s}$ equal to the fraction $\frac{10001}{10000}$, the difference of whose terms is 1, and their sum 20001; therefore in this case, $z=\frac{1}{20001}$, but it will be sufficient for our purpose to make $z=\frac{1}{20000}$; then since A or the first term of the series is $2pz$, we shall have Az^2 or $B=2pz^3$, and $\frac{B}{3}$ or the second term of the series $=\frac{2pz^3}{3}=\frac{p}{12000000000000}=.00000000000003$: here then it is plain that the second term of the series, when it is greatest, amounts not to above 3 in the fourteenth place; but generally speaking, it makes not it's entry till the fifteenth or sixteenth place; for certainly this term $\frac{2pz^3}{3}$ will be greater or less in the same proportion as z^3 is greater or less; and therefore if in any case z be but the tenth part of what it is in this, the second term there will be but the thousandth part of what it is here.

This being allowed, let the number whose logarithm is required be 1234556789: here having put down 9 for the characteristic of the logarithm sought, I point the number proposed thus, 12345.56789, making the five first places integral, and the rest, be they fewer or more, decimal; see art. 393 paragraph 2: then making the integral part 12345 equal to s, and the decimal part $.56789=t$, the logarithm of $\frac{s+t}{s}$ will be $\frac{2pt}{2s+t}$, as in the lemma; and this logarithm added to the logarithm of s taken out of the *canon*, will give the logarithm of $s+t$ required.

Let us now try whether we cannot reduce this logarithm $\frac{2pt}{2s+t}$ to a more simple expression: in order to which, let c be the logarithm of $\frac{s}{s-1}$.

$\frac{s}{s-1}$, d the logarithm of $\frac{s+1}{s}$, and make $c-d=e$; then c will be the difference of the logarithms of $s-1$ and s, d will be the difference of the logarithms of s and $s+1$, and e will be what is called the second difference of the three logarithms of $s-1$, s and $s+1$; the former two, to wit c and d, are expressed in the *canon*, and this last e will be the difference of those two differences. From these definitions of c, d and e, and from the foregoing lemma, I draw the following equations, to wit, $c=\frac{2p}{2s-1}$, $d=\frac{2p}{2s+1}$, and $c-d$ or $e=\frac{4p}{4ss-1}$. Now since $d=\frac{2p}{2s+1}$, we shall have $2p=\overline{2s+1}\times d$, and $\frac{4p}{4ss-1}$ or $e=\frac{\overline{2s+1}\times 2d}{4ss-1}$; divide both the numerator and denominator of this last fraction by $2s+1$, and it will be reduced to $\frac{2d}{2s-1}$; therefore $e=\frac{2d}{2s-1}$. Again, since $2p=\overline{2s+1}\times d$ as above, we shall have $\frac{2pt}{2s+t}$ or the logarithm of $\frac{s+t}{s}$, equal to $\frac{2s+1}{2s+t}\times dt$: let v be the complement of t to unity, that is, let $t+v=1$; then you will have $\frac{2s+1}{2s+t}=\frac{2s+t+v}{2s+t}=1+\frac{v}{2s+t}$; whence $\frac{2s+1}{2s+t}\times dt=dt+\frac{dtv}{2s+t}$; so that the logarithm of $\frac{s+t}{s}$ is now found equal to $dt+\frac{dtv}{2s+t}$: instead of $2s+t$ in the denominator of this last fraction write $2s-1$, and the error when greatest will not amount to so much as 4 in the fourteenth place, as will best be seen after what will be delivered in the next article; then instead of $\frac{dtv}{2s+t}$ you will have $\frac{dtv}{2s-1}$, or $\frac{1}{2}etv$, since $\frac{2d}{2s-1}=e$ as above; and the logarithm of $\frac{s+t}{s}$ will be found at last equal to $dt+\frac{1}{2}etv$.

This being obtained, I shall in the next place find the proportional part, in order to compare this logarithm with it. I say therefore, as 1 the difference of the numbers s and $s+1$, is to d the difference of their logarithms, so is t the difference of the numbers s and $s+t$, to dt a proportional part of the difference of their logarithms; by which we learn, that of the two parts dt and $\frac{1}{2}etv$, which compose the logarithm of $\frac{s+t}{s}$, the first and most considerable part dt is the proportional part,

 and

and therefore the other $\frac{1}{2}etv$ may be called the correction of the proportional part.

Let us now return to the example formerly proposed, which was to find the logarithm of the number 123456789. Here then by the *canon* we find the decimal part of the logarithm belonging to the five first places equal to .0914910943, $c=.0000351812$, $d=.0000351783$, $c-d$ or $e=.0000000029$; whence $dt=.0000199774$I: but in correcting the proportional part, there will be no occasion for more than two places of the number t; therefore in this case I make $t=.57$, and consequently $1-t$ or $v=.43$; whence $tv=.245$, and $\frac{etv}{2}=.0000000003\,5$ +; this correction added to .0000199774I the proportional part before found, gives the logarithm of $\frac{s+t}{s}$ (as far as can be expressed by ten decimal places) equal to .0000199778; and this last logarithm being added to 9.0914910943, the logarithm of 123450000, gives the logarithm of 123456789 equal to 9.0915110721.

Note. These operations may be contracted by Mr. *Oughtred*'s rule, which I would have the reader learn at his leisure, and practise as occasion requires. See *Oughtred*'s *clavis*, of multiplication contracted.

Now if we would apply this rule for finding the proportional part dt, we must observe, that 3, the last figure of d, is supposed to be in the tenth decimal place, the index of which place according to Mr. *Oughtred*, is -10; therefore to have eleven decimal places as nearly as they can be obtained, I place 5, whose index is -1, under 3, whose index is -10, that the last figure in the product towards the right hand may be in the eleventh decimal place. See the work.

$$
\begin{array}{r l}
.0000351783 & =d \\
98765. & =t \text{ inverted} \\
\hline
1758915 & \\
211070 & \\
24625 & \\
2814 & \\
317 & \\
\hline
.0000199774I & =dt
\end{array}
\qquad\qquad
\begin{array}{r l}
.57 & =t \\
34. & =v \text{ inverted} \\
\hline
228 & \\
17 & \\
\hline
.245 & =tv \\
9200000000. & =e \text{ inverted} \\
\hline
49 & \\
22 & \\
\hline
.0000000007I & =etv.
\end{array}
$$

Whence $\frac{etv}{2}=.0000000003\,5$ +.

405. In the last article, the second difference e was found to be $\frac{4p}{4ss-1}=\frac{p}{ss}$ very nearly: whence we may conclude that this second difference will be greatest when s is least, that is, when s is equal to 10000; substitute therefore 10000 instead of s in the value $\frac{p}{ss}$, and you will have the greatest second difference that can arise equal to $\frac{p}{100000000}$ $=.0000000043$. Again, if instead of $\frac{1}{2}e$ in the correction $\frac{1}{2}etv$, be substituted it's value $\frac{p}{2ss}$, the correction will then be equal to $\frac{ptv}{2ss}$: whence it follows that this correction will be greatest when the product tv is greatest, and at the same time the number s is least, that is, when $tv=\frac{1}{4}$, and $s=10000$: for the sum of t and v being always equal to unity, their product tv will be the greatest when t and v are each equal to $\frac{1}{2}$, as will easily appear upon examination: substitute therefore $\frac{1}{4}$ instead of tv, and 10000 instead of s in the value of the correction $\frac{ptv}{2ss}$, and you will have the greatest correction that can happen, equal to $\frac{p}{800000000}=.00000000054$; for though the logarithms of this *canon* do not extend to above ten places, yet it will be proper to compute both the proportional part and it's correction to eleven places, in order to make sure of ten.

From what has been delivered in the last article, it may be further observed, that towards the latter end of the scale it is not impossible but that two numbers differing from one another only by an unit in the tenth and last place, may have one and the same logarithm in *Vlacq*'s *canon*, that is, the ten first places of both their logarithms may be the same, how much soever they may differ afterwards. As for instance, the logarithm of this number 10000000000 is 10 .0000000000, and the logarithm of this fraction $\frac{10000000000}{10000000000-1}$ is $\frac{2p}{20000000000-1}$ $=.00000000004$; therefore if 10 .0000000000 be the true logarithm of 10000000000, it will also be the nearest logarithm of 10000000000 -1; that is, no whole number with ten decimal figures annexed can express it nearer.

Proposition 8.

406. *To find to ten places the absolute number belonging to any logarithm in* Vlacq*'s canon magnus.*

Retaining

Retaining the notation of the two laſt articles, and ſetting aſide the characteriſtic, take out the neareſt tabular logarithm leſs than the logarithm propoſed, and the abſolute number over againſt it will be s, part of the number ſought. Now to find the other part t, ſubtract the ſaid tabular logarithm from the logarithm propoſed, and call the remainder r; then will r be the logarithm of the fraction $\frac{s+t}{s}$; but the logarithm of $\frac{s+t}{s}$ was found in the laſt article but one to be $dt+\frac{dtv}{2s+t}$; therefore $dt+\frac{dtv}{2s+t}=r$; whence $t=\frac{r}{d}-\frac{tv}{2s+t}$. But as d the difference of the logarithms of the two numbers s and $s+1$, is to 1 the difference of the numbers themſelves, ſo is r, which is but part of the former difference, to $\frac{r}{d}$ a proportional part of the latter; therefore of the reſidual expreſſing the quantity t, that is, of the reſidual $\frac{r}{d}-\frac{tv}{2s+t}$ the affirmative and main part $\frac{r}{d}$ may be called the proportional part, and the negative part $\frac{tv}{2s+t}$ or $\frac{tv}{2s}$ may be called the correction to be ſubtracted; which correction can never exceed .0000125, as may eaſily be tried by ſubſtituting $\frac{1}{4}$ for tv, and 10000 for s, as in the laſt article: therefore the proportional part $\frac{r}{d}$ expreſſes the quantity t, either exactly, or if there be any error, it will not be above 1 too much in the fifth and laſt place.

Now to find the correction, let t be groſſly expreſſed by the two firſt places of $\frac{r}{d}$; and then ſubtracting it from unity, the remainder will be the two firſt places of the quantity v: divide now two or three of the firſt places of the product tv by two or three of the firſt places of the number $2s$, and the quotient will be a correction more than ſufficient; which being ſubtracted from $\frac{r}{d}$ will leave t as correct as five decimal places can expreſs it.

N. B. By the tabular logarithms here I mean the logarithms of all numbers from 10000 to 100000, the logarithms of all numbers under 10000 being included in theſe.

EXAMPLE

EXAMPLE.

Setting aside the characteristic, let it be required to find the absolute number belonging to this logarithm .0915110721: the next less tabular logarithm is .0914910943, over against the absolute number 12345; therefore $s=12345$. Again, the tabular logarithm subtracted from the logarithm proposed leaves $.0000199778=r$, and $d=.0000351783$, as in the table; therefore $\frac{r}{d}=.56790$, the two first places whereof give $t=.57$; whence $1-t$ or $v=.43$; whence $tv=.245+$; call it .25, and then dividing it by 25000, the two first places of $2s$, the quotient will be .00001, which is the correction; subtract this from $\frac{r}{d}$, that is, from .56790, and there remains .56789; therefore $t=.56789$; and the digits representing the absolute number sought will be 1234556789, which number must be ordered as the characteristic directs; as if the characteristic be 8, the last figure must be a decimal; if 9, the whole must be integral; if 10, the number must be raised one degree higher by a cypher to the right hand, &c.

BOOK IX. PART III.

Of the invention of Divisors.

HITHERTO I have considered no sorts of equations, but only simple and quadratic: but intending now to proceed to others of higher forms, I shall endeavour to prepare the learner for them by acquainting him with such things as may be of any use to him in the resolution of those equations.

First then I shall begin with the invention of divisors: and because in this, as in almost every thing else, our Countreyman Sir *Isaac Newton* has considerably outdone all those who have written upon the same subject before him, I shall give the learner his rules, with my own explication of them wherever they seem to want it; but shall not think it necessary to pursue that matter so far as Sir *Isaac* has done.

Before I proceed, the reader is to understand, that *By the invention of divisors is meant, the finding out of all those quantities by which any given quantity, simple or compound, may be divided without a remainder.*

407. *If the quantity whose divisors are sought be a simple quantity; that is, if it be only the product of the multiplication of others, and is not made up of parts connected together by the sign + or —, divide it by it's least divisor, and then the quotient by it's least divisor, and so on till you come to a quotient that is not further divisible: by this means you will have all the prime numbers or divisors by which the quantity proposed can be divided. Then multiplying these prime divisors two and two together, make out as many different products as you can this way: then make out as many more products as you can by multiplying them three and three together, and so on; and these products thus found will be all the composite divisors the given quantity will admit of.*

As for example, let it be required to assign all the divisors of the number 60. First I divide 60 by it's least divisor 2, and then the quotient 30 by it's least divisor 2, then the quotient 15 by 3; and so come at last to the quotient 5, which is not capable of any further division: thus then I have obtained all the prime divisors of the number 60, to wit, 1, 2, 2, 3, 5; or leaving out 1, because it is a common divisor of all numbers, 2, 2, 3, 5. Then I make out of these numbers as many different couples as I can, multiplying them together, and so have other divisors, as 2×2, 2×3, 2×5, 3×5, that is, 4, 6, 10, 15: then I multiply them three and three together, and so have other divisors, as $2\times2\times3$, $2\times2\times5$, $2\times3\times5$, that is, 12, 20, 30: lastly I multiply them all four together, and so have the divisor $2\times2\times3\times5$, or 60.

Again, if all the divisors be required which will divide the number $21abb$ without a remainder; divide the number $21abb$ by 3, and the quotient $7abb$ by 7, and the quotient abb by a, and the quotient bb by b, and you will come at last to the prime quotient b; therefore the prime divisors of the number $21abb$ are 3, 7, a, b, b: these multiplied two and two together give 21, $3a$, $3b$, $7a$, $7b$, ab, bb: multiply them three and three together, and you will have $21a$, $21b$, $3ab$, $3bb$, $7ab$, $7bb$, abb: multiply them four and four together, and you will have $21ab$, $21bb$, $3abb$, $7abb$: multiply all five together, and you will have the quantity itself that was first proposed, to wit $21abb$.

In like manner all the divisors of the number $2abb-6aac$ may be found by first dividing by 2, and then the quotient $abb-3aac$ by a; after which you will come to the indivisible quotient $bb-3ac$; therefore the divisors of the quantity $2abb-6aac$ are 2, a, $bb-3ac$; $2a$, $2bb-6ac$, $abb-3aac$; $2abb-6aac$.

N. B. *Numbers singly considered are called prime numbers, when they will admit of no divisors less than themselves, except unity.* Thus 2, 3, 5 and 7 are called prime numbers; when 4, 6, 9 and 10 are called composite numbers, as being produced by the multiplication of others.

408.

408. That all the numbers found as in the laſt article will divide the quantities there propoſed, appears ſo evident at firſt ſight as to need no demonſtration: but that thoſe quantities will not admit of any other diviſors than ſuch as are obtained by the foregoing method, will not perhaps be ſo obvious to every one; this therefore I ſhall endeavour to evince by the following demonſtration.

Let abc be any number whoſe prime diviſors according to the laſt article are a, b and c, and conſequently whoſe compoſite diviſors are ab, ac, bc and abc; I ſay then that this number will admit of no other diviſors beſides theſe, either prime or compoſite. And firſt I ſay it will admit of no other prime diviſors: for let d be any prime number different from any of the prime numbers a, b and c; then ſince a and d are two unequal prime numbers, they muſt neceſſarily be prime to each other; and the ſame may be obſerved of the numbers b and d, as alſo of the numbers c and d. Since then the numbers a and b are both prime to d, their product ab will be prime to d, by art. 195: in like manner, ſince the number ab and the number c are both prime to d, their product abc will alſo be prime to d; therefore d can be no diviſor of the number abc; and the ſame will be true of any other prime number whatever that is different from the numbers a, b and c. I ſay in the next place, that the number abc will admit of no other compoſite diviſors beſides thoſe already ſpecified; for we have aſſigned all the compoſite diviſors that can poſſibly be formed out of the prime diviſors a, b and c; and we have alſo ſhewn that there are no other prime diviſors from which any other compoſite diviſors can be derived. *Q. E. D.*

N. B. That the numbers a and b found as above directed are prime numbers as well as the number c (which is ſo by the ſuppoſition) will be evident from hence; that if a was not a prime number, it would admit of ſome diviſor as e, leſs than itſelf: ſince then e divides a, and a divides the number abc, e would alſo divide the number abc; whence a would not be the leaſt diviſor of that number, contrary to the ſuppoſition; therefore a muſt be ſome prime number; and the ſame may alſo be proved of b, which is (by the ſuppoſition) the leaſt diviſor of the firſt quotient bc.

409. *If a quantity, after having been divided by all it's ſimple diviſors, remains ſtill a compound quantity, and there be any ſuſpicion that it will admit of ſome compound diviſor, range it's ſeveral members according to the dimenſions of ſome letter in it; then ſubſtitute for that letter three or more terms of this arithmetical progreſſion,* 3, 2, 1, 0, −1, −2; *which done, place the numbers reſulting from theſe poſitions over againſt them, together with all their diviſors both affirmative and negative; or if the negative diviſors be not put down, they are at leaſt to be underſtood: laſtly, overagainſt*

 th-

these divisors put down all the decreasing arithmetical progressions that can be made out of them, running from the highest to the lowest rank, as do the terms of the progression 3, 2, 1, 0, —1, —2; *but the difference of the terms of the progressions made out of the divisors must either be unity, or some other number that will divide the highest power of the dividend proposed: if any such progression can be found, take that term which stands against* 0 *in the first progression,* 1, 0, —1 &c, *and dividing it by the common difference of the progression, join the quotient with it's proper sign to the letter according to the dimensions whereof the dividend was disposed, and you will have a quantity with which a division is to be tried.*

As for example, let the quantity whose compound divisor is sought be $x^3-xx-10x+6$. Here substituting the terms 1, 0, —1 for x successively, I find the numbers resulting from these positions to be —4, 6, 14; these therefore without regarding their signs, I place together with their divisors, over against the respective terms of the progression from whence they arise, in the manner following:

	1, 2, 4.	+4,
6	1, 2, 3, 6.	+3,
14	1, 2, 7, 14.	+2.

Then because the highest power of the dividend, to wit x^3, admits of no numeral divisor but unity, I look amongst the divisors to find an arithmetical progression whose common difference is unity, and whose terms decrease after the same manner as the terms of the first progression 1, 0, —1; and of this sort of progression I find amongst the divisors but one, to wit, 4, 3, 2; the number 4 being one of the divisors that stand overagainst 1, the number 3 one of those overagainst 0, and the number 2 one of those overagainst —1: of this progression 4, 3, 2, I take the number 3 that stands overagainst 0, and joining it with the letter x, I try a division with the quantity $x+3$, and find it succeeds; for $x^3-xx-10x+6$ being divided by $x+3$ quotes $xx-4x+2$ without any remainder.

Again, let the quantity whose compound divisor is sought be $6y^4-y^3-21y^2+3y+20$. Here putting 1, 0, —1 successively for y, the numbers resulting are 7, 20, 3, which together with their divisors I put down overagainst the terms 1, 0, —1 from whence they arise, in the manner following:

1	7	1, 7.	7, 7,	1, —1,
0	20	1, 2, 4, 5, 10, 20.	4, 5,	—1, —2,
—1	3	1, 3.	1. 3.	—3. —3.

Then examining the divisors, I find amongst them four decreasing arithmetical

metical progressions, to wit, 7, 4, 1 whose common difference is 3; 7, 5, 3 whose common difference is 2; 1, —1, —3 whose common difference is also 2, and —1, —2, —3 whose common difference is 1. Since then all these common differences will divide 6, the coefficient of the highest term, I try with them all one after another thus: I first begin with the first progression 7, 4, 1, and dividing 4, which stands overagainst o in the first progression 1, o, —1, by 3, the common difference of the terms of the progression, the quotient is $\frac{4}{3}$; this I join with the letter y, and so try a division with $y+\frac{4}{3}$, or (which is all one) with $3y+4$, and it succeeds; for if $6y^4—y^3—21yy+3y+20$ be divided by $3y+4$, the quotient will be $2y^3—3yy—3y+5$ without a remainder. Then I consider the next progression 7, 5, 3, and dividing 5, overagainst o, by 2, the common difference of the progression, I try a division with $y+\frac{5}{2}$ or (which is all one) with $2y+5$; but the operation does not succeed. In the third place I try with the next progression 1, —1, —3, and dividing —1, overagainst o, by 2, the common difference of the progression, I try a division with $y—\frac{1}{2}$, or (which is the same thing) with $2y—1$; but neither does this operation succeed. In the last place I try with the last progression —1, —2, —3, attempting a division with $y—2$; but neither does this succeed: so that upon the whole matter, the quantity proposed admits of but one compound divisor, to wit $3y+4$.

Lastly let the quantity proposed be $24a^5—50a^4+49a^3—140a^2+64a+30$. Here substituting 2, 1, o, —1 successively for a (because the dimensions of the quantity proposed run pretty high,) the numbers resulting are —42, —23, +30, —297; these, without any regard to their signs, I put down with their divisors overagainst the terms of the progression as follows:

2	42	1, 2, 3, 6, 7, 14, 21, 42.	+3, +3, +7,
1	23	1, 23.	+1, —1, +1,
o	30	1, 2, 3, 5, 6, 10, 15, 30.	—1, —5, —5,
1	297	1, 3, 9, 11, 27, 33, 99, 297.	—3. —9. —11.

Then examining these divisors, I find three, decreasing arithmetical progressions, to wit 1st, 3, 1, —1, —3; 2d, 3, —1, —5, —9; 3d, 7, 1, —5, —11: of these three progressions I take the terms overagainst o, to wit —1, —5, —5, and dividing them by the common differences of their respective progressions, to wit 2, 4, 6, I have three quantities to try with, namely $a—\frac{1}{2}$, $a—\frac{5}{4}$, and $a—\frac{5}{6}$, whereof only the last $a—\frac{5}{6}$ or $6a—5$ succeeds, the quotient being $4a^4—5a^3+4a^2—20a—6$.

If no divisor can be discovered this way, we may safely conclude that the quantity proposed admits of no compound divisor of one dimension: but not-

 withstanding

withstanding all this, it may still have a divisor where the indeterminate quantity arises to two or more dimensions, as will presently be shewn. But first I suppose it will be expected that I give some explication of the method already described, which shall be done in the next article.

410. If the quantity $x^3-xx-10x+6$ be divided by $x+3$, the quotient will be $xx-4x+2$ as in the first example, without any remainder; and as there is nothing in this division that restrains x to any one signification rather than another, but that it is left absolutely indeterminate, it is evident that the conclusion must be universal; that is, if x be put equal to any whole number whatever, the number $x+3$ will be a divisor of the number $x^3-xx-10x+6$, and the quotient will be x^2-4x+2. As for instance, make $x=10$; then you will have the dividend $x^3-xx-10x+6=1000-100-100+6=806$, and the divisor $x+3=10+3=13$, and the quotient $xx-4x+2=100-40+2=62$; therefore 13 is a divisor of the number 806; and if the number 806 be divided by 13, the quotient will be 62; both which upon tryal will be found to be true.

This being observed, in order to prepare the reader for what follows, let the quantity ax^3+bx^2+cx+d admit of some compound divisor of one dimension, which we will suppose to be $ex+f$, and let the quotient be gx^2+hx+k, the signs $+$ signifying no more than that the quantities to which they are prefixed are to be added to the foregoing according to the common rules of addition, whether they be affirmative or negative: then it is plain that the quantities ax^3+bx^2+cx+d must constitute a whole number, as also the quantities $ex+f$ another whole number, let the value of x be what it will, provided it be a whole number. This (I say) is plain; for it is only upon this supposition that the rule in the last article can have any place; therefore a, b, c, d, e, f must be whole numbers; for if any of them were fractions, such values might be ascribed to x as would make the numbers constituted by them fractions. It is evident also from the supposition, that the quotient $gxx+hx+k$ must be a whole number, what whole number soever x is made to represent; for to suppose, that in all cases of x, the whole number $ex+f$ is a divisor of the whole number ax^3+bx^2 &c, is to suppose the quotient a whole number; for otherwise any number whatsoever may be said to be a divisor of any other; therefore g, h, k are whole numbers.

Let us now suppose $x=1$, and we shall have $ax^3+bx^2+cx+d=a+b+c+d$, and $ex+f=e+f$; therefore if this resulting number $a+b+c+d$, together with it's divisors, be placed overagainst 1, according to the rule of the last article, the number $e+f$ will be one of those divisors.

Again,

Again, let $x=0$; then we shall have $ax^3+bx^2+cx+d=d$, and $ex+f=f$; therefore if d with it's divisors be placed overagainst 0, the number f will be one of them.

Lastly, let $x=-1$; then we shall have $ax^3+bx^2+cx+d=-a+b-c+d$, and $ex+f=-e+f$; therefore if the number $-a+b-c+d$ together with it's divisors be placed overagainst -1, the number $-e+f$ will be found amongst them: but the divisors $e+f$, f, and $-e+f$ form a decreasing arithmetical progression, the common difference whereof e, is the number by which x must be multiplied to make ex part of the divisor sought, and whereof the term f, standing overagainst 0, being added to the part ex, will make $ex+f$ a compleat divisor: therefore if ax^3+bx^2+cx+d admits of any compound divisor $ex+f$ of one dimension, there must occur such a progression as we have here described; though it does not follow *e converso*, that wherever there occurs such a progression, the quantity proposed will admit of a compound divisor of this kind. But to avoid (as far as possible) all fruitless enquiries, the author of this rule excludes all such progressions whose common difference will not divide a, the coefficient of the highest term ax^3, and that very justly; for if ax^3 &c be divided by ex &c, the first term of the quotient will be $\frac{a}{e} \times xx = gxx$; whence $\frac{a}{e} = g$ a whole number; therefore e must be a divisor of a.

If it be demanded why the author, after having obtained the quantities e and f, considers the divisor first as $x+\frac{f}{e}$ before he reduces it to $ex+f$, the reason is plain; for it is possible that f may be divisible by e, in which case $\frac{f}{e}$ will be a whole number, as l; or if not so, yet $\frac{f}{e}$ may sometimes be reducible to less terms, as $\frac{l}{m}$; and whichsoever of these cases happens, the divisor $x+l$ in the former case, or $mx+l$ in the latter, will either of them be a more simple divisor than $ex+f$. These cases will happen when the practitioner uses not the simplest progressions that are to be found amongst the divisors, that is, progressions whose terms admit of no common divisor.

If it be objected, that a quantity may admit of a divisor of this form $-ex+f$, and that the author's rule, that prescribes only decreasing progressions, will not find it; I answer, that whatever quantity can be divided by $-ex+f$, can also be divided by it's contrary $+ex-f$; and this is a divisor which the author's rule will find wherever it exists.

411.

411. *If the quantity proposed rises not to above three dimensions, and admits of a compound divisor of two, it must necessarily have another divisor of one dimension, which being found by the foregoing rule, and dividing the quantity proposed, the quotient will be the other divisor: but if it rises to above three dimensions, and hath no divisor of less than two, such a divisor must be found by the following method.*

Let x *be the undeterminate quantity of whose powers the quantity proposed consists; then substituting the terms of this progression* 3, 2, 1, 0, —1, —2 *successively for* x, *and putting down overagainst them the numbers resulting, together with all their affirmative divisors, as before, the negative ones being understood; multiply the squares of the terms of the progression by some numeral divisor of the highest term of the quantity proposed, which we will call* e, *and put down the products* 9e, 4e, 1e, 0, 1e, 4e *overagainst the terms of the progression. This done, subtract these products from all the divisors of their respective ranks, both affirmative and negative, and put down the remainders in the same ranks under their proper signs; then try if in passing from the highest to the lowest rank of remainders, you can find an arithmetical progression of any sort, whether increasing or decreasing; if one or more such progressions can be found, put them down overagainst the terms of the first progression, and then examine them one after another thus: let (in any of these progressions)* ±g *be the term that stands overagainst* 0 *in the first progression; then subtracting this term* ±g *from the next above it in the same progression, let the remainder be* ±f; *I say then that the compound quantity with which the division is to be tried, will be* exx ±fx ±g. *But if no divisor can be found this way, try another divisor of the coefficient of the highest term of the quantity proposed for* e, *and so proceed on till you have passed over all the affirmative numeral divisors of the highest term; and then if no compound divisor can be found, it will be an infallible argument that the quantity proposed admits of none.*

One example will be sufficient to illustrate this rule. Let then the quantity proposed be $x^4-x^3-5xx+12x-6$. Here substituting the numbers 3, 2, 1, 0, —1, —2 successively for x, I find the numbers resulting to be 39, 6, 1, —6, —21, —26; these, without any regard to their signs, together with their divisors, I put down as usual; then multiplying the squares of the terms of the progression by 1, which is the only affirmative numeral divisor of x^4, the highest term of the quantity proposed, and putting down the products 9, 4, 1, 0, 1, 4 overagainst the terms of the progression, I subtract 9 from all the divisors of it's rank, both affirmative and negative, that is, from 39, 13, 3, 1, —1, —3, —13, —39, and the remainders, to wit 30, 4, —6, —8, —10, —12, —22, —48, I place overagainst 9 in the highest rank: the same I do with the next number 4, with respect to the divisors of it's rank; and

and so of all the rest. Then I examine the remainders, and find among them two arithmetical progressions passing from the highest to the lowest rank, to wit, 4, 2, 0, —2, —4, —6, and —6, —3, 0, +3, +6, +9, both which I put down overagainst the terms of the first progression 3, 2, 1, 0, —1, —2. In the former of these two progressions I find the term —2 standing overagainst 0 in the first progression; therefore $g=-2$; this subtracted from 0, the next term above it in the same progression, gives +2 for f: hence, since $e=1$, the divisor to be tried, drawn from this progression, will be $xx+2x-2$. In the other progression I find the term overagainst 0 to be 3; therefore $g=3$; and this subtracted from 0 above, leaves —3 for f; therefore the divisor from this progression, by which the division is to be tried, is $xx-3x+3$. I try them both, and they both succeed; that is, if the quantity proposed be divided by either of these divisors, the quotient will be the other.

3	39	39, 13, 3, 1.	9	30,	4,	—6,	—8,	—10,	—12,	—22,	—48.	4,	—6,
2	6	6, 3, 2, 1.	4	2,	—1,	—2,	—3,	—5,	—6,	—7,	—10.	2,	—3,
1	1	1.	1	0,	—2.							0,	0,
0	6	6, 3, 2, 1.	0	6,	3,	2,	1,	—1,	—2,	—3,	—6.	—2,	3,
—1	21	21, 7, 3, 1.	1	20,	6,	2,	0,	—2,	—4,	—8,	—22.	—4,	6,
—2	26	26, 13, 2, 1.	4	22,	9,	—2,	—3,	—5,	—6,	—17,	—30.	—6.	9

These operations may (when they want it) be contracted a little thus: let p, q, r, s, t, v *be the numbers resulting from the positions* 3, 2, 1, 0, —1, —2, *and as such let them be put down overagainst them; then if the extreme numbers* p *and* v *be pretty large, or contain many divisors, let these divisors, as well as the remainders arising from them, be omitted, and try if you can find an arithmetical progression passing through the remainders belonging to the intermediate numbers* q, r, s, t: *if such a one can be found, let the terms be* 2f+g, f+g, g, —f+g; *then it is plain that in this progression, the next term above* 2f+g *will be* 3f+g, *and the next below* —f+g *will be* —2f+g: *add the number* 9e *to the term* 3f+g, *and try whether the number* 9e+3f+g *will divide the number* p; *if it does, then add also* 4e *to* —2f+g, *and try whether the number* 4e—2f+g *will divide the number* v: *if this division also succeeds, you will then have as full a progression to try with, as if the divisors and remainders of the extreme numbers* p *and* v *had been put down: but if neither, or but one of these divisions succeeds, it is an argument that the progression thus found was accidental, and did not spring from any compound divisor.* All this is manifest from the rule already laid down.

If

If the coefficient of the higheſt term of the quantity propoſed admits of many diviſors, they may create the Analyſt ſome trouble; but in moſt caſes where this invention of diviſors is of any uſe, I mean in ſinking equations from higher to lower forms, this coefficient will generally be unity, or may eaſily be made ſo if it happens to be otherwiſe, as will be ſhewn upon another occaſion.

N. B. In the directions here given I have taken the liberty to vary from my author in a circumſtance or two of no moment as to the concluſion.

412. After what has been delivered in the laſt article but one, little needs be ſaid in explication of the laſt. For let any quantity have a diviſor of this form ex^2+fx+g; then it is plain that e, the coefficient of the higheſt term of the diviſor, muſt be ſome numeral diviſor of the higheſt term of the quantity propoſed, from what was proved in art. 410. It is evident alſo that if x be made equal to 3, the diviſor ex^2+fx+g will be $9e+3f+g$; if $x=2$, the diviſor will be $4e+2f+g$, and ſo on: therefore amongſt the ſeveral diviſors ſtanding overagainſt the terms of the progreſſion 3, 2, 1, 0, -1, -2, there will be found theſe in particular, running from the higheſt to the loweſt rank, to wit, $9e+3f+g$, $4e+2f+g$, $e+f+g$, g, $e-f+g$, $4e-2f+g$; therefore after the numbers $9e$, $4e$, $1e$, 0, $1e$, $4e$ are ſubtracted from all the diviſors of their reſpective ranks, there muſt be found running through the remainders, this arithmetic progreſſion, $3f+g$, $2f+g$, $f+g$, g, $-f+g$, $-2f+g$: therefore *e converſo*, whenever ſuch a progreſſion occurs, it ought to be tried whether it will furniſh out a compound diviſor or not.

If f be negative, that is, if the diviſor be of this form, $ex^2-fx\pm g$, the progreſſion produced by it will be an increaſing arithmetical progreſſion, as $-3f\pm g$, $-2f\pm g$, $-f\pm g$, $\pm g$, $+f\pm g$, $+2f\pm g$.

Laſtly, if f be equal to nothing, that is, if the diviſor be of this form, $exx\pm g$, the terms of the progreſſion (if in this caſe I may call it ſo) will be all equal to $\pm g$, and to one another.

413. *If the quantity propoſed be made out of the powers of two different letters, ſo as to have all it's terms of the ſame number of dimenſions, inſtead of one of the letters put unity; then finding (by the foregoing rules) a compound diviſor, if any ſuch there be, fill up the deficient dimenſions of the diviſor by thoſe of the letter that was before ſuppreſſed, and you will have the diviſor compleated.*

As for inſtance, let the quantity propoſed be $6y^4-ay^3-21a^2y^2+3a^3y+20a^4$, every term whereof conſiſts of four dimenſions, either of the letter y, or of the letter a, or of both together. Here then ſubſtituting 1 for a, the former quantity will be changed into this, $6y^4-y^3-21y^2+3y+20$;

$+3y+20$, whereof $3y+4$ is a compound divisor found in art. 409 Now as $3y$, the first term of this divisor, hath one dimension of the letter y, and as the other term 4 hath no dimension of any letter, I supply one dimension from the letter a, and so make the divisor $3y+4a$.

Again, let the quantity proposed be $x^4-ax^3-5a^2x^2+12a^3x-6a^4$: this quantity, substituting 1 for a, becomes $x^4-x^3-5x^2+12x-6$, whereof $xx+2x-2$ is a compound divisor found in art. 411: supply the deficient dimensions of this divisor from those of the letter a, so that every term may have two dimensions as well as the first, and it will be $xx+2ax-2aa$.

414. No one can be ignorant of the reason of the rule given in the last article, but he who is too lazy to try it; for if he will give himself the trouble to divide, first the quantity $6y^4-y^3-21y^2+3y+20$ by $3y+4$, and then the quantity $6y^4-ay^3-21a^2y^2+3a^3y+20a^4$ by $3y+4a$, he will easily see that the numeral parts of both divisions are and must be the same; which may also be observed of the quantities belonging to the second example.

As there are but few quantities that admit of compound divisors in comparison of those that do not; if the learner has a mind to exercise himself in more examples of this kind besides those we have here given, the best way will be for him, first to assume some compound quantity, and then to multiply it into any other of what form he pleases, provided it does not ascend to above two dimensions; for then he may assure himself that the product when obtained will have at least one compound divisor, to wit the multiplicator.

There are some other cases wherein compound divisors may be discovered; but they are so very troublesome, and happen so very rarely, that I think they will scarce justify my insisting any longer upon this subject.

BOOK IX. PART IV.

Of the Arithmetic of surd quantities.

HAVING as yet but slightly touched upon the doctrine of irrational or surd quantities as occasion required, and being now obliged to enter more deeply into such parts of it as are of any use in the resolution of cubic and biquadratic equations, I shall take this opportunity to give it the reader entire and in one view, that he may know where to have it whenever he has occasion for it.

Note. $\sqrt{a}$ or $\sqrt[2]{a}$ signifies the square root of a; $\sqrt[3]{a}$ or $\sqrt[c]{a}$ signifies the third or cube root of a; $\sqrt[4]{a}$ the fourth or biquadrate root of a, that is, such a quantity whose square squared is equal to a.

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415. *A rational number is that, whose ratio or proportion to unity can be expressed in finite numbers; and so comprehends all whole numbers, mixt numbers, and proper fractions:* thus $2\frac{3}{4}$ is a rational number, because $2\frac{3}{4}$ is to 1 as 11 to 4. *Therefore an irrational number is that whose ratio to unity cannot be expressed in finite numbers:* thus if the diameter of a circle be represented by unity or 1, the circumference must be represented by an irrational number; because the ratio of the circumference to the diameter cannot be expressed by finite numbers: thus $\sqrt{2}$ is an irrational number, because no numbers are subtil enough to express it's ratio to unity, as hath been amply shewn in another part of this treatise. *Now this last irrational number, to wit $\sqrt{2}$, and all others involving the roots of numbers that are not to be extracted, are particularly called Surds;* and it is of this sort of quantity we are now to treat.

To reduce surds of different roots to others of the same root.

416. This was done in art. 379, paragraph 8, and I shall here do it again after another manner, thus. Let it be required to reduce the square root of 2 and the cube root of 3 to equal surds of the same root: now as the index of the square root is 2, and that of the cube root 3, and as the least common multiple of 2 and 3 is 6, I raise both roots to the sixth power, thus: put x for $\sqrt{2}$, and y for $\sqrt[3]{3}$; then we shall have $xx=2$, and $x^6=8$; whence $x=\sqrt[6]{8}$: again, as $y=\sqrt[3]{3}$, we have $y^3=3$, and $y^6=9$, and $y=\sqrt[6]{9}$; therefore the square root of 2, and the cube root of 3 are the same with the sixth root of 8, and the sixth root of 9 respectively. *Universally thus; let* $\sqrt[m]{a}$ *and* $\sqrt[n]{b}$ *be required to be reduced: to do this, I reduce the fraction* $\frac{m}{n}$ *to it's least terms, and in that state call it* $\frac{r}{s}$; *then will* ms, *which is equal to* nr, *be the least common multiple of the two indexes* m *and* n, *as in art.* 171, *corol.* 2: *let us call this common multiple* p, *and put* x *for* $\sqrt[m]{a}$; *then we shall have* $x^m=a$;

$x^m = a$; *raise both sides of the equation to the power of* $\frac{p}{m}$, *which is done by multiplying both indexes by the index* $\frac{p}{m}$, *and you will have* $x^p = a^{\frac{p}{m}}$: *but* $p = ms$, *as above, and* $\frac{p}{m} = s$: *therefore* $x^p = a^s$, *and* $x = \sqrt[p]{a^s}$. *In like manner if* y *be put for* $\sqrt[n]{b}$, *we shall have* $y^n = b$, *and* $y^p = b^{\frac{p}{n}} = b^r$; *whence* $y = \sqrt[p]{b^r}$: *so that the surds* $\sqrt[m]{a}$ *and* $\sqrt[n]{b}$, *when reduced to the same root, are equal to* $\sqrt[p]{a^s}$ *and* $\sqrt[p]{b^r}$ *respectively.*

Of the addition and subtraction of surd quantities.

417. *Surd quantities of different denominations, generally speaking, are no more capable of addition or subtraction than algebraic quantities are: thus the sum of* $\sqrt{3}$ *and* $\sqrt{2}$ *is* $\sqrt{3}+\sqrt{2}$, *and their difference* $\sqrt{3}-\sqrt{2}$. *But surds of the same denomination may be added or subtracted as easily as any other quantities that are of the same denomination: thus* $3\sqrt{10}+2\sqrt{10} = 5\sqrt{10}$, *and* $3\sqrt{10}-2\sqrt{10} = \sqrt{10}$. *And universally, if two surds of the same root are such, as that, setting aside the common radical sign, one is to the other as any one square number to another when they involve the square root, or as any one cube number to another when they involve the cube root,* &c; *I say then that these two surds are capable of being reduced into one, either by addition or subtraction as occasion requires. As if* a *and* b *signify two quantities, whereof* a *is the greater, and whereof* a *is to* b *as* c^2 *to* d^2; *I say then that* $\sqrt{a}$ *and* $\sqrt{b}$ *are capable of addition and subtraction: for since* a *is to* b *as* c^2 *to* d^2, *we have* $a = \frac{c^2}{d^2} \times b$, *and* $\sqrt{a} = \frac{c}{d} \times \sqrt{b}$, *and* $\sqrt{a}+\sqrt{b} = \frac{c+d}{d} \times \sqrt{b}$, *and* $\sqrt{a}-\sqrt{b} = \frac{c-d}{d} \times \sqrt{b}$. Thus $\sqrt{8}+\sqrt{2} = 3\sqrt{2}$, and $\sqrt{8}-\sqrt{2} = \sqrt{2}$; for setting aside the common radical sign, 8 is to 2 as 4 is to 1, that is as one square number to another; and therefore in this case $c = 2$, $d = 1$, $\frac{c+d}{d} = 3$, $\frac{c-d}{d} = 1$. Thus $\sqrt{18}+\sqrt{8} = \frac{5}{2}\sqrt{8}$, and $\sqrt{18}-\sqrt{8} = \frac{1}{2}\sqrt{8}$.

Of the multiplication and division of surd quantities.

418. *Whenever one surd quantity is to be multiplied or divided by another, reduce them first to the same root (by the last article but one) if they be*

be not of the same root already; then setting aside the common radical sign, multiply or divide one quantity by the other, and prefix the common radical sign to the product or quotient. Thus $\sqrt{a}\times\sqrt{b}$ gives $\sqrt{ab}$, $\frac{\sqrt{a}}{\sqrt{b}}=\sqrt{\frac{a}{b}}$, $\sqrt[3]{a}\times\sqrt[3]{b}=\sqrt[3]{ab}$, $\frac{\sqrt[3]{a}}{\sqrt[3]{b}}=\sqrt[3]{\frac{a}{b}}$, &c; all which I thus demonstrate. Make $\sqrt{a}=x$, and $\sqrt{b}=y$; then we shall have $x^2=a$, and $y^2=b$, and $x^2y^2=ab$, and $xy=\sqrt{ab}$, that is, $\sqrt{a}\times\sqrt{b}=\sqrt{ab}$: we have also $\frac{x^2}{y^2}=\frac{a}{b}$, and $\frac{x}{y}=\sqrt{\frac{a}{b}}$, that is, $\frac{\sqrt{a}}{\sqrt{b}}=\sqrt{\frac{a}{b}}$. Again, if $x=\sqrt[3]{a}$, and $y=\sqrt[3]{b}$, we shall have $x^3=a$, and $y^3=b$, and $x^3y^3=ab$, and $\frac{x^3}{y^3}=\frac{a}{b}$; whence $xy=\sqrt[3]{ab}$, and $\frac{x}{y}=\sqrt[3]{\frac{a}{b}}$.

Other examples of this sort of multiplication and division are as follows.

1. $\sqrt[3]{a}\times\sqrt[3]{a}=\sqrt[3]{a^2}$.
2. $a\sqrt{b}$ or $a\times\sqrt{b}=\sqrt{a^2}\times\sqrt{b}=\sqrt{a^2b}$.
3. $\frac{a}{\sqrt{b}}=\frac{\sqrt{a^2}}{\sqrt{b}}=\sqrt{\frac{a^2}{b}}$.
4. $\frac{\sqrt{a}}{b}=\frac{\sqrt{a}}{\sqrt{b^2}}=\sqrt{\frac{a}{b^2}}$.
5. $a\sqrt{b}\times c\sqrt{d}=ac\times\sqrt{bd}$.
6. $\frac{a\sqrt{b}}{c\sqrt{d}}=\frac{a}{c}\times\sqrt{\frac{b}{d}}$.
7. $a\sqrt{b}\times c\sqrt{b}=abc$.
8. $\frac{a\sqrt{b}}{c\sqrt{b}}=\frac{a}{c}$
9. $\sqrt{a}\times\sqrt[3]{a}=\sqrt[6]{a^3}\times\sqrt[6]{a^2}=\sqrt[6]{a^5}$.
10. $\frac{\sqrt{a}}{\sqrt[3]{a}}=\frac{\sqrt[6]{a^3}}{\sqrt[6]{a^2}}=\sqrt[6]{a}$.
11. $\sqrt{a}\times\sqrt{b}\times\sqrt{c}=\sqrt{abc}$.
12. $\sqrt{8}\times\sqrt{2}=\sqrt{16}=4$.
13. $\frac{\sqrt{8}}{\sqrt{2}}=\sqrt{4}=2$.
14. $\sqrt[4]{9}\times\sqrt[4]{9}=\sqrt[4]{81}=\sqrt[2]{9}=3$.
15. $\overline{2+\sqrt{3}}\times\overline{2-\sqrt{3}}=4 \quad 1$.
16. The square of $\sqrt{2}+\sqrt{3}=2+3+2\sqrt{6}=5+2\sqrt{6}$.
17. The cube of $\sqrt{2}+\sqrt{3}=2\sqrt{2}+6\sqrt{3}+9\sqrt{2}+3\sqrt{3}=11\sqrt{2}+9\sqrt{3}$.

To reduce roots from higher to lower denominations.

419. *This is done by dividing both the index of the root, and the index of the power whereof it is the root by their greatest common measure, and so using the new indexes instead of those from whence they were derived.* Thus, $\sqrt{a^2}$,

$\sqrt[4]{a^2}=\sqrt[2]{a}$: thus $\sqrt[12]{a^{15}}=\sqrt[4]{a^5}$; for make $x=\sqrt[12]{a^{15}}$, then you will have $x^{12}=a^{15}$; extract the cube root of both sides, since both the indexes 12 and 15 can be divided by 3, and you will have $x^4=a^5$; whence $x=\sqrt[4]{a^5}$.

To reduce surds, when possible, from higher to lower terms.

420. *This is done by taking off the radical sign, and then resolving the number to which it was prefixed into it's prime factors or divisors by art. 407; for if any of these factors be twice or thrice repeated, or oftener, the number to which the radical sign was prefixed may be resolved into the product of a square and non-square number when the square root is concerned, or into the product of a cube and non-cube number when the cube root is concerned,* &c; *and so it's square or cube root will be reduced to the square or cube root of a lesser quantity multiplied into some rational number.* As if $\sqrt{7350}$ be proposed; this number $7350=2\times3\times5\times5\times7\times7$, whereof $5\times5\times7\times7$ is a square number, whose root is $5\times7=35$; and the other part 2×3 or 6 is a non-square; therefore $7350=35\times35\times6$, and $\sqrt{7350}=35\sqrt{6}$. Again, let $\sqrt{288}$ be proposed; this number $288=2\times2\times2\times2\times2\times3\times3$, whereof $2\times2\times2\times2\times3\times3$ is a square number, whose root is $2\times2\times3$ or 12, and the remaining factor 2 is a non-square; therefore $\sqrt{288}=12\sqrt{2}$. Lastly, let $\sqrt[3]{96}$ be proposed; this number $96=2\times2\times2\times2\times2\times3$, whereof $2\times2\times2$ is a cube number, whose side or root is 2, and the other part $2\times2\times3$ or 12 is a non-cube; therefore $\sqrt[3]{96}=2\sqrt[3]{12}$.

To free the denominator of a fraction from all radicality, where the square or biquadrate root, or both are concerned.

421. This will best be shewn by examples, which let be these that follow.

Let the denominator of any fraction be $\sqrt{b}$; then if both the numerator and denominator of that fraction be multiplied by $\sqrt{b}$, you will have the denominator equal to b a rational number. In like manner if the denominator of a fraction be $\sqrt[4]{a+b}$, multiply both it's terms by $\sqrt[4]{a+b}$, and you will have the denominator equal to $\sqrt[2]{a+b}$; multiply now both terms by $\sqrt[2]{a+b}$, and you will have the denominator equal to $a+b$ a rational number.

Let the denominator be $\sqrt{a}+\sqrt{b}$; then multiply both terms by $\sqrt{a}-\sqrt{b}$, and you will have the denominator equal to $a-b$ a rational number.

Let the denominator be $\sqrt{a}+\sqrt{b}+\sqrt{c}$; multiply both terms by $\sqrt{a}+\sqrt{b}-\sqrt{c}$, and you will have the denominator equal to $a+b-c+2\sqrt{ab}$: make $a+b-c=d$, so that the denominator may be $d+2\sqrt{ab}$; then if both terms be multiplied by $d-2\sqrt{ab}$, you will have the denominator equal to $dd-4ab$ a rational number.

Lastly, let the denominator be $\sqrt{a}+\sqrt[4]{b}$; multiply both terms by $\sqrt{a}-\sqrt[4]{b}$, and you will have the denominator equal to $a-\sqrt[4]{bb}=a-\sqrt{b}$; multiply again by $a+\sqrt{b}$, and you will have the denominator equal to $aa-b$ a rational number.

To extract the square root of any binomial, whose two parts when squared are commensurable to each other.

422. Note, *that under the name of a binomial I here, as well as in other places, understand two quantities connected together by + or —, though the latter by some be called a residual.*

Let the binomial whose square root is required be $a\pm b$; *then it is plain that* a *must be the greater part*; for if it was otherwise, $a-b$ would be a negative number, and could have no square root: let $x+y$ be the square root of $a+b$, and let $x-y$ or $y-x$, according as x is greater or less than y, be the square root of $a-b$; then we shall have $xx+2xy+yy=a+b$, and $xx-2xy+yy=a-b$; and these two equations will be the same, whether x represents the greater or the lesser part of the root sought; so that the same operation that gives one part must give the other also. Add and subtract the equation $x^2-2xy+y^2=a-b$ to and from the equation $x^2+2xy+y^2=a+b$, and the resulting equations will be $2x^2+2y^2=2a$, and $4xy=2b$; the former gives $yy=a-x^2$, and the latter gives $yy=\frac{bb}{4xx}$; therefore $a-xx=\frac{bb}{4xx}$, and $axx-x^4=\frac{bb}{4}$; and changing the signs, $x^4-ax^2=\frac{-b^2}{4}$; and compleating the square, $x^4-ax^2+\frac{aa}{4}=\frac{aa-bb}{4}$: *substitute* ss *instead of* $aa-bb$, *and* you will have $x^4-ax^2+\frac{aa}{4}=\frac{ss}{4}$; extract the square root of both sides, and you will have $xx-\frac{a}{2}=\pm\frac{s}{2}$; whence $xx=\frac{a\pm s}{2}$; that is, the square of the greater

part

part will be $\frac{a+s}{2}$, and the ſquare of the leſſer part will be $\frac{a-s}{2}$; therefore *the binomial root ſought will be* $\sqrt{\frac{a+s}{2}} \pm \sqrt{\frac{a-s}{2}}$, *according as* b *is affirmative or negative.*

And here it muſt be obſerved, that *If* a^2 *and* b^2 *be commenſurable to each other, one may be expreſſed in parts of the other; and ſo the difference* a^2-b^2, *and conſequently* s, *will be but one ſimple quantity, rational if the difference of the ſquares of* a *and* b *happens to be a ſquare number, and irrational if otherwiſe: but if* a^2 *and* b^2 *be incommenſurable to each other, the root ſought will be more embarraſſed, and will not come down to that degree of ſimplicity which the intent of this problem is to find out wherever it is to be met with.*

The ſynthetical demonſtration of the foregoing theorem is as follows. We are to prove that $\sqrt{\frac{a+s}{2}} \pm \sqrt{\frac{a-s}{2}}$ is the ſquare root of the binomial $a \pm b$; or (which amounts to the ſame) we are to prove that the ſquare of the former is equal to the latter. Now the ſquare of the ſum or difference of any two numbers is found by increaſing or diminiſhing the ſum of their ſquares by their double product: but the ſquare of $\sqrt{\frac{a+s}{2}}$ is $\frac{a+s}{2}$, and the ſquare of $\sqrt{\frac{a-s}{2}}$ is $\frac{a-s}{2}$, and therefore the ſum of the ſquares of the two parts of this binomial root is a. Again, $\sqrt{\frac{a+s}{2}} \times \sqrt{\frac{a-s}{2}}$ gives $\sqrt{\frac{aa-ss}{4}}$; but $aa-bb=ss$, and therefore $aa-ss=bb$, and $\frac{aa-ss}{4}=\frac{bb}{4}$; therefore $\sqrt{\frac{aa-ss}{4}}$, or the rectangle of the two parts, equals $\frac{b}{2}$, and the double rectangle is b. Increaſe now or diminiſh a, the ſum of the ſquares above found, by b, the double product of the parts, and you will have the ſquare of $\sqrt{\frac{a+s}{2}} \pm \sqrt{\frac{a-s}{2}} = a \pm b$. *Q. E. D.*

EXAMPLE I.

Let the ſquare root of this binomial $3 \pm \sqrt{8}$ be required. Here $a=3$, $b=\pm\sqrt{8}$, a^2-b^2 or $s^2=9-8=1$, $s=1$, $\frac{a+s}{2}$ (or the ſquare of the greater part) $=2$, $\frac{a-s}{2}$ (or the ſquare of the leſſer part) $=1$; whence the

the root fought is $\sqrt{2}\pm1$, as will further appear upon tryal; for the ſquare of $\sqrt{2}\pm1$ is $2+1\pm2\sqrt{2}=3\pm\sqrt{8}$.

EXAMPLE 2.

Let the ſquare root of $5\pm\sqrt{24}$ be required. Here $a=5$, $b=\pm\sqrt{24}$, $s=1$, $\frac{a+s}{2}=3$, $\frac{a-s}{2}=2$; whence the root fought is $\sqrt{3}\pm\sqrt{2}$. But here perhaps it may be objected, that certainly the moſt direct way of extracting the ſquare root of the binomial $5\pm\sqrt{24}$ would be to extract, as far as occaſion requires, the ſquare root of 24, and then again the ſquare root of $5+\sqrt{24}$ or $5-\sqrt{24}$; whereas after the extraction of the root this way is over, we have ſtill two roots to extract, to wit $\sqrt{3}$ and $\sqrt{2}$, before we can come to the true quantity of the root required. But whoſoever makes this objection, does not ſeem to enter into the true deſign of this *analyſis*, which is not ſo much to come at the true quantity of the root required, as it is to enquire into the nature and conſtitution of the binomial propoſed; that is, what rational or irrational quantities are the ingredients that enter into it's compoſition.

EXAMPLE 3.

Let the ſquare root of this binomial, to wit, $\sqrt{32}\pm\sqrt{24}$ be required. Here then $a=\sqrt{32}$, $b=\pm\sqrt{24}$, $s=\sqrt{8}$, $\frac{a+s}{2}=\frac{\sqrt{32}+\sqrt{8}}{2}$, $\frac{a-s}{2}=\frac{\sqrt{32}-\sqrt{8}}{2}$: but $\sqrt{32}=4\sqrt{2}$, and $\sqrt{8}=2\sqrt{2}$; therefore $\frac{\sqrt{32}+\sqrt{8}}{2}=3\sqrt{2}=\sqrt{18}$, and $\frac{\sqrt{32}-\sqrt{8}}{2}=\sqrt{2}$: therefore the ſquare of the greater part is $\sqrt{18}$, and the ſquare of the leſſer part is $\sqrt{2}$, and the ſquare root required is $\sqrt[4]{18}\pm\sqrt[4]{2}$.

EXAMPLE 4.

Let the ſquare root of this quantity be required, to wit, $rr\pm2x\times\sqrt{r^2-x^2}$, whereof r^2 is one part, and $2x\times\sqrt{r^2-x^2}$ is the other. Here I make $a=r^2$, and $b=\pm2x\times\sqrt{r^2-x^2}$; whence $a^2=r^4$, $b^2=4r^2x^2-4x^4$, a^2-b^2 or $s^2=r^4-4r^2x^2+4x^4$, $s=r^2-2x^2$, $\frac{a+s}{2}=r^2-x^2$, $\frac{a-s}{2}=x^2$; therefore r^2-x^2 is the ſquare of one of the parts ſought, and x^2 is the ſquare of the other; ſo that the root required is $\sqrt{r^2-x^2}\pm x$.

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EXAMPLE 5.

Let the ſquare root of this quantity be required, to wit, $\frac{v+\sqrt{v^2-4z^2}}{2}$, or (which is the ſame thing in the preſent caſe) let it be required to find the two parts ſeparately which compoſe the ſquare root of the aforeſaid quantity. Here $a=\frac{v}{2}$, $b=\frac{\sqrt{v^2-4z^2}}{2}$, $b^2=\frac{v^2-4z^2}{4}$, a^2-b^2 or s^2 $=\frac{4z^2}{4}=z^2$, $s=z$, $a+s=\frac{v}{2}+z=\frac{v+2z}{2}$, $\frac{a+s}{2}$ or the ſquare of the greater part equals $\frac{v+2z}{4}$, $\frac{a-s}{2}$ or the ſquare of the leſſer part equals $\frac{v-2z}{4}$; whence the greater part is $\frac{1}{2}\sqrt{v+2z}$, the leſſer part $\frac{1}{2}\sqrt{v-2z}$, and the whole $\frac{1}{2}\sqrt{v+2z}+\frac{1}{2}\sqrt{v-2z}$. In like manner, had the ſquare root of the quantity $\frac{v-\sqrt{v^2-4z^2}}{2}$ been required, it would have been found to be $\frac{1}{2}\sqrt{v+2z}-\frac{1}{2}\sqrt{v-2z}$.

EXAMPLE 6.

Laſtly, let the quantity whoſe ſquare root is required be $6+\sqrt{8}-\sqrt{12}-\sqrt{24}$. To turn this quantity into a binomial, I make the affirmative part $6+\sqrt{8}=a$, and the negative part $-\sqrt{12}-\sqrt{24}=-b$; then I have $a^2=36+8+12\sqrt{8}=44+12\sqrt{8}$, $b^2=+12+24+2\sqrt{288}$: but $24+12=36$, and $2\sqrt{288}=24\sqrt{2}=12\sqrt{8}$; therefore $b^2=36+12\sqrt{8}$; ſubtract this from a^2, that is from $44+12\sqrt{8}$, and you will have a^2-b^2 or $s^2=8$, and $s=\sqrt{8}$; therefore $\frac{a+s}{2}=3+\sqrt{8}$, and $\frac{a-s}{2}$ $=3$; therefore the root required is $\sqrt{3+\sqrt{8}}-\sqrt{3}$, to wit $-\sqrt{3}$ becauſe the ſign of b was negative: but $\sqrt{3+\sqrt{8}}$ was found in the firſt example to be $1+\sqrt{2}$; therefore the root ſought comes out at laſt $1+\sqrt{2}-\sqrt{3}$.

When more radical quantities than one are concerned in the given ſquare, their number muſt be either $1+2$, *that is* 3, *as in the laſt example, or* $1+2+3$, *that is* 6, *or* $1+2+3+4$, *that is* 10, &c, as will eaſily appear from the next following article.

But here the great *Newton* according to his wonted ſagacity makes an obſervation, which (though obvious enough when made) perhaps but few

few besides himself could have hit upon; it is this: *When more roots than one are concerned in the given square, and these are all incommensurable, so as not to be reducible to fewer terms, the root sought will consist of more than two parts; and these parts will best be found thus: multiply any two of these radicals together, and divide the product by such a one of the remaining radicals as will make a rational quotient, and the square root of half this quotient will be one of the parts sought; and so you may proceed till you have got as many different quotients as can be obtained this way: the number of different quotients will be equal to the number of parts in the root sought, and every quotient will give a correspondent part. Thus in the last example,* $\frac{\sqrt{8}\times\sqrt{12}}{\sqrt{24}}=\sqrt{4}=2$, $\frac{\sqrt{8}\times\sqrt{24}}{\sqrt{12}}=4$, $\frac{\sqrt{12}\times\sqrt{24}}{\sqrt{8}}=6$; *therefore all the different quotients that can be obtained are* 2, 4, 6, *whose halves are* 1, 2, 3, *whose square roots are* 1, $\sqrt{2}$, $\sqrt{3}$, *the three parts of the root sought. Now as to the signs of these parts, they may be thus determined: the only affirmative radical quantity in the given square is* $\sqrt{8}$, *which includes two parts of the root sought, to wit,* 1 *and* $\sqrt{2}$; *therefore the signs of those two parts* 1 *and* $\sqrt{2}$ *must be both alike: the next radical* $-\sqrt{12}$ *is negative, and includes the parts* 1 *and* $\sqrt{3}$; *therefore the signs of those two parts* 1 *and* $\sqrt{3}$ *must be unlike: therefore the root required is* $1+\sqrt{2}-\sqrt{3}$, *or* $-1-\sqrt{2}+\sqrt{3}$, *either of which will equally express the root of the given square.*

423. In order to account for this last rule taken out of *Newton*'s Algebra, I must demonstrate the following lemma.

Let a *and* b *be two irrational incommensurable quantities, but such whose squares* a^2 *and* b^2 *are both rational: I say then that* ab *the product of their multiplication will be irrational.* For a is incommensurable to b *ex hypothesi*; therefore ab is incommensurable to b^2; but b^2 is rational; therefore ab is irrational. *Q. E. D.*

This being allowed, let there be a trinomial, as $a+b+c$, the squares of whose parts are all rational numbers; but let the parts themselves be either all, or all but one, irrational; let them also be incommensurable to one another, so as to be incapable of being contracted into fewer terms: then if we square this trinomial, that by examining the composition we may be the better able to form a resolution, we shall find the square to be $aa+2ab+2ac+bb+2bc+cc$, whereof the parts $aa+bb+cc$ compose a rational number, such as was the number 6 in the last example; but the other three, to wit $2ab$, $2ac$, $2bc$ will be all irrational by the lemma, such as were the radicals $\sqrt{8}$, $\sqrt{12}$, $\sqrt{24}$ in the last example. Now whosoever considers these irrational parts may easily observe, that every part of the square includes two parts of the root, that the product of any two parts of the square includes all the three parts of the root,

and

and consequently that if any of these products be divided by the remaining part, which takes in two parts of the root, there will be but one single part left in the quotient. Thus if $2ab$ be multiplied into $2ac$, the product will be $4aabc$, which being divided by the remaining part $2bc$, will have $2aa$ for the quotient, half whereof is aa, and the square root of that half is a, which is one of the parts of the root sought; and so of the rest.

When there are more than three radicals in the given square, it is not impossible but that sometimes two irrational parts may be multiplied together which cannot be divided by any of the remaining parts; but there will always be as many different quotients obtainable as are equal in number to the number of parts in the root sought, and consequently as many as are sufficient: thus if the quadrinomial $a+b+c+d$ be squared, the square will be $aa+2ab+2ac+2ad+bb+2bc+2bd+cc+2cd+dd$; where setting aside the rational parts, all the other parts being six in number $(3+2+1)$ will be irrational; that is, $2ab$, $2ac$, $2ad$; $2bc$, $2bd$; $2cd$: now if of these six parts $2ab$ and $2cd$, or $2ac$ and $2bd$, or $2ad$ and $2bc$ be multiplied together, the product in all the three cases will be $4abcd$, which cannot be divided by any of the remaining parts so as to have a rational quotient: but these are only three combinations out of fifteen wherein the division will not succeed, the other twelve giving four different quotients three times repeated, answerable to the four parts of the quadrinomial root $a+b+c+d$.

424. The only use I shall make at present of the foregoing *analysis*, is in the resolution of such biquadratic equations as fall under the name and form of quadratics. As for example, let the following equation be proposed to be resolved, to wit, $6x^2-x^4=1$: here changing the signs, you have $x^4-6x^2=-1$, and compleating the square, $x^4-6x^2+9=8$, and extracting the square root, $xx-3=\pm\sqrt{8}$; whence $xx=3\pm\sqrt{8}$, and $x=\pm\sqrt{3\pm\sqrt{8}}$; but $+\sqrt{3\pm\sqrt{8}}$ is $\sqrt{2}\pm1$, by the first example of the last article but one; therefore $-\sqrt{3\pm\sqrt{8}}$ will be $-\sqrt{2}\mp1$; whence $x=+\sqrt{2}\pm1$, or $-\sqrt{2}\mp1$; that is, the four roots of the foregoing quadratic equation will be $+\sqrt{2}+1$, $+\sqrt{2}-1$, $-\sqrt{2}+1$, and $-\sqrt{2}-1$; and any of these being substituted for x in the original equation, will answer the condition of the equation. As for instance, let x be made equal to $\sqrt{2}+1$; then we shall have $x^2=2+1+2\sqrt{2}=3+\sqrt{8}$; whence $6xx=18+6\sqrt{8}$: again, since $xx=3+\sqrt{8}$, we shall have $x^4=9+8+6\sqrt{8}=17+6\sqrt{8}$: subtract now x^4 from $6xx$, and the remainder will be 1, as the equation requires: and the same will be the case of all the other roots of the equation.

Again, let the equation proposed be $xx\times\sqrt{128}-x^4=8$, or $x^4-xx\times\sqrt{128}=-8$. Here the coefficient of the second term is $-\sqrt{128}$, whose half is $-\frac{\sqrt{128}}{2}=-\frac{\sqrt{128}}{\sqrt{4}}=-\sqrt{32}$; therefore compleating the square, you have $x^4-xx\times\sqrt{128}+32=24$; extract the square root and you will have $xx-\sqrt{32}=\pm\sqrt{24}$, and $xx=\sqrt{32}\pm\sqrt{24}$: put s for $\sqrt{32}\pm\sqrt{24}$, and you will have $x=\pm\sqrt{s}$; but by the third example of the last article but one, $+\sqrt{s}=\sqrt[4]{18}\pm\sqrt[4]{2}$, and therefore $-\sqrt{s}=-\sqrt[4]{18}\mp\sqrt[4]{2}$; therefore $x=+\sqrt[4]{18}\pm\sqrt[4]{2}$, or $-\sqrt[4]{18}\mp\sqrt[4]{2}$.

A LEMMA.

425. *If a binomial whose parts when squared are both rational, be raised to a cube, and this cube be resolved into another binomial in such a manner as shall presently be shewn; I say then that the two parts of the cube will be affected with the same surds as the two correspondent parts of the root, and no other; comparing the greater part of the cube with the greater part of the root, and the lesser part of the cube with the lesser part of the root: and* vice versa.

For of the two quantities x and y whose squares xx and yy are both rational, let x be greater than y; then will $x-y$ be affirmative; and so will it's cube, that is, $x^3-3xxy+3xyy-y^3$ will be an affirmative quantity; therefore x^3+3xyy will be greater than y^3+3xxy; but $x^3+3xyy=x\times\overline{xx+3yy}$, and $y^3+3xxy=y\times\overline{yy+3xx}$; therefore the cube of $x-y$ may be resolved into this binomial, to wit, $x\times\overline{xx+3yy}-y\times\overline{yy+3xx}$, whereof $x\times\overline{xx+3yy}$ is the greater part: but as $\overline{xx+3yy}$ and $\overline{yy+3xx}$ are both rational quantities by the supposition, it is certain that $x\times\overline{xx+3yy}$ the greater part of the cube, can have no surd in it but what is included in x the greater part of the root; and that $y\times\overline{yy+3xx}$ the lesser part of the cube can have no surd in it but what is included in y the lesser part of the root; and *vice versa*: so that if the surds included either in the cube or in the cube root, be known, those in the other will be known too. *Q. E. D.*

To extract the third, fifth or seventh root &c of a binomial whose parts, when squared, are both rational.

426. Before I enter upon this *analysis*, give me leave to add to the foregoing lemma another observation of no less importance to our subject, which

which is, that *If in any case I can find the cube root of a multiple of any quantity, it will be the same thing as if I had the cube root of the quantity itself.* For let dz be a multiple of d, and let dz be also a cube whose cube root is c; then will $c^3=dz$, and $\frac{c^3}{z}=d$, and $\frac{c}{\sqrt[3]{z}}=\sqrt[3]{d}$.

These things being observed, let it now be required to extract the cube root of the following quantity, if it admits of any such root, to wit $a\pm b$, supposing a to be the greater part, and b the less.

Now that I may be the less confined in the solution of this problem, I shall enquire, not so much into the cube root of $a\pm b$ as of $\overline{a\pm b}\times z$, leaving z undetermined till I can fix such a value upon it as will best serve my purpose. Let then $x+y$ be the cube root of $az+bz$, and $x-y$ that of $az-bz$; and since a is greater than b by the supposition, that is, since $az-bz$ is affirmative, it's cube root $x-y$ will be so too, and x will be the greater part of the cube root sought. Again, since $x+y$ is the cube root of $\overline{a+b}\times z$, and $x-y$ that of $\overline{a-b}\times z$, we shall have (by the rule of surd multiplication) $\overline{x+y}\times\overline{x-y}$, that is $xx-yy$, the cube root of $\overline{a^2-b^2}\times z^2$. Now to the end that x^2-y^2 may be known, let us assume for zz any rational number, square or no square it matters not, which multiplying a^2-b^2 will make it a cube. This may always be done; for should other numbers fail, I can make z equal to a^2-b^2, and so can make $\overline{a^2-b^2}\times z^2$ equal to a cube whose root is a^2-b^2; but the less number zz stands for, the better: let then $\overline{a^2-b^2}\times z^2$ be a rational cube whose side is n, and you will have $xx-yy=n$. Having proceeded thus far, let r be a quantity nearly expressing the cube root of $\overline{a+b}\times z$, which may be taken in gross, and we shall have $x+y=r$ nearly; and since $\overline{x+y}\times\overline{x-y}$, that is, $r\times\overline{x-y}=n$, we shall have $x-y=\frac{n}{r}$ nearly: add now these two equations together, to wit $x+y=r$ and $x-y=\frac{n}{r}$, and you will have $2x=r+\frac{n}{r}$ nearly, that is, more nearly than either $x+y$ was equal to r, or $x-y$ was equal to $\frac{n}{r}$; for if r be taken too much, $\frac{n}{r}$ will be too little, and their sum $r+\frac{n}{r}$ will be an interme-

diate

diate quantity: ſince then $2x=r+\frac{n}{r}$, we ſhall have $x=\frac{r+\frac{n}{r}}{2}$; and thus we have got the value of x pretty nearly, and may be as exact in it as we pleaſe; but we are ſtill ignorant of it's compoſition, which here is the thing chiefly wanted, and therefore this is what we muſt next enquire into.

Now it is certain by the foregoing lemma, that x the greater part of the root muſt have included in it the ſame ſurd as az the greater part of the cube; therefore if the quantity az be reduced to it's leaſt terms by art. 420, and whatever is rational in it be thrown away, ſo that there remains only one ſurd, which will be the leaſt that is included in az, and which we ſhall call s; the ſame ſurd s will alſo be included in x, that is, x will be equal to s multiplied into ſome rational number: call this rational number t, that is, let $ts=x=\frac{r+\frac{n}{r}}{2}$, and you will have $t=\frac{r+\frac{n}{r}}{2s}$: take then the neareſt whole number, or the neareſt ſimple fraction, if that approaches nearer to the quantity $\frac{r+\frac{n}{r}}{2s}$, and you will have the coefficient t, and conſequently ts or x, which will be accurate if the quantity propoſed admits of a cube root.

Having thus got ts or x, the greater part of the root ſought, the other part will be eaſily obtained: for ſince $xx-yy$, that is $ttss-yy=n$, we ſhall have $y=\sqrt{t^2s^2-n}$, and the whole cube root of the binomial $\overline{a\pm b}\times z$ (if it has any) will be $ts\pm\sqrt{t^2s^2-n}$; whence the cube root of the binomial $a\pm b$ will be $\frac{ts\pm\sqrt{t^2s^2-n}}{\sqrt[3]{z}}$, to wit, $+$ or $-\sqrt{t^2s^2-n}$, according as b the leſſer part of the binomial is affirmative or negative. Raiſe now this binomial to a cube; and if it then be found equal to $a\pm b$, you have what you wanted; if otherwiſe, you may reaſonably conclude that the quantity propoſed admits of no binomial cube root.

As the greater part of the root was here found independently of the leſſer, ſo alſo may the leſſer part be found independently of the greater thus. From the equation $x+y=r$ ſubtract the equation $x-y=\frac{n}{r}$, and you will have $2y=r-\frac{n}{r}$; whence $y=\frac{r-\frac{n}{r}}{2}$: let s be the leaſt ſurd included in bz the leſſer part of the cube, and make $ts=y$; then will

t be

t be the nearest whole number or the nearest simple fraction to $\frac{r-\frac{n}{r}}{2s}$, and so ts or y will be known; whence x may easily be found, being equal to $\sqrt{t^2s^2+n}$, and the binomial to be tried will be $\frac{\pm ts+\sqrt{t^2s^2+n}}{\sqrt[3]{z}}$. Thus may either part be found independently of the other; and it is proper the learner should know both these ways, because it very often happens that one part or the other of the given cube is rational, and then s in that part will be equal to unity, and so the correspondent part of the root will be the more easily obtained.

From what has been said, the rules for extracting the cube root of the binomial $a\pm b$ (if it admits of a binomial root) are these that follow.

1. *Take* zz *any rational number, square or not square, but the less the better, and such, that* $\overline{aa-bb}\times zz$ *may be a rational cube. If no lesser number can be found,* z *may be taken equal to* $aa-bb$, *and* $\overline{aa-bb}\times zz$ *will be a cube: let it's cube root be* n. *If* $aa-bb$ *be itself a cube,* zz *(and consequently* z*) will be* 1.

2. *Then take a whole number or a simple fraction, nearly equal to the cube root of* $\overline{a+b}\times z$, *and call it* r.

3. *This done, there are two ways of finding the cube root of the quantity proposed, to wit, either by* az *or* bz. *The former way is this: if* az *be a surd, divide the number under the radical sign by the greatest square included in it, and the quotient with the radical sign prefixed (which I call* s*) will be the least surd included in* az: *but if* az *be rational,* s *will be a unit.*

4. *Take then the nearest whole number, or the nearest simple fraction (if it approaches nearer) to the quantity* $\frac{r+\frac{n}{r}}{2s}$, *and call it* t: *I say that the cube root of the binomial* $a\pm b$ *(if it admits of a binomial cube root, which must be tried by cubing the root when found) will be* $\frac{ts\pm\sqrt{ttss-n}}{\sqrt[3]{z}}$, *according as the quantity proposed was* $a+b$ *or* $a-b$.

5. *The other way is, by finding* s *the least surd included in* bz, *if* bz *be a surd quantity, (or if it be rational, then taking* $s=1$,*) and making* $t=\frac{r-\frac{n}{r}}{2s}$, *and the cube root of* $a\pm b$ *(to be tried as above mentioned) will be* $\frac{\sqrt{ttss+n}\pm ts}{\sqrt[3]{z}}$.

Note.

Note. If az be rational, it may be convenient to work by the third and fourth rules, but if bz be rational, by the fifth, whereby (s being a unit) the cube root will be more easily found.

As to the fifth and seventh roots, these are extracted just in the same manner as the third root; except that in these cases zz *must be so assumed, as multiplying* aa—bb, *the product may have a rational fifth or seventh root, which must now be called* n; r *must be the fifth or seventh root of* $\overline{a+b}\times z$, *and instead of* $\sqrt[3]{z}$ *in the denominator of the binomial root to be tried, must be substituted* $\sqrt[5]{z}$, $\sqrt[7]{z}$, &c. See *Newton*'s method of extracting the third, fifth and seventh root &c of a binomial in his reduction of radicals, where our zz answers to his Q.

N. B. The fourth root is obtained by extracting the square root twice, the sixth root by extracting first the square and then the cube root, the eighth root by extracting the square root thrice, the ninth root by extracting the cube root twice, &c.

EXAMPLE I.

Let it be required to extract the cube root out of the following quantity, to wit, $\sqrt{980}+\sqrt{972}$. Here then $a=\sqrt{980}$, $b=\sqrt{972}$, $a^2-b^2=980-972=8$, therefore in this case a^2-b^2 without multiplication, is itself a rational cube, whose side is 2; therefore $n=2$, $z^2=1$, $z=1$, and $\sqrt[3]{z}=1$. Again, $\sqrt{980}=31+$, and $\sqrt{972}=31+$; therefore $\overline{a+b}\times z=\overline{31+31}\times 1=62$, whose cube root is $4-$; therefore $r=4$, and $r+\frac{n}{r}=4\frac{1}{2}$. Lastly, $980=2\times2\times5\times7\times7=14\times14\times5$; therefore $\sqrt{980}$ or $az=14\sqrt{5}$; therefore $s=\sqrt{5}=2\frac{1}{4}$ nearly; therefore $\frac{r+\frac{n}{r}}{2s}=\frac{4\frac{1}{2}}{4\frac{1}{2}}=1$; therefore $t=1$, $ts=\sqrt{5}$, $t^2s^2-n=5-2=3$, and $\sqrt{t^2s^2-n}=\sqrt{3}$; therefore $\frac{ts+\sqrt{t^2s^2-n}}{\sqrt[3]{z}}$, or the root to be tried, is $\sqrt{5}+\sqrt{3}$, and it succeeds; for the cube of $\sqrt{5}+\sqrt{3}$ is $5\sqrt{5}+15\sqrt{3}+9\sqrt{5}+3\sqrt{3}=14\sqrt{5}+18\sqrt{3}=\sqrt{980}+\sqrt{972}$.

EXAMPLE 2.

Let it be required to extract the cube root of $\sqrt{968}-25$. Here $a=\sqrt{968}$, $b=25$, $a^2-b^2=968-625=343=7\times7\times7$; therefore $n=7$, $z=1$.

$z=1$. Again, $\sqrt{968}=31+$; therefore $\overline{a+b}\times z=\overline{31+25}\times 1=56$, whose cube root is $4-$; therefore $r=4$, and $r-\frac{n}{r}=4-\frac{7}{4}=\frac{9}{4}$, for I here intend to find the lesser part of the root, as being rational; therefore in this case $s=1$, and $\frac{r-\frac{n}{r}}{2s}=\frac{9}{8}=1$ nearly; therefore I make t and consequently $ts=1$; whence $\sqrt{t^2s^2+n}=\sqrt{8}$; therefore the binomial root to be tried is $\sqrt{8}-1$, and it succeeds.

EXAMPLE 3.

Let the cube root of $9-\sqrt{80}$ be required. Here $a=9$, $b=\sqrt{80}$; $a^2-b^2=1$, $n=1$, $z=1$; and as $\sqrt{80}=9-$, we shall have $\overline{a+b}\times z=18$, whose cube root lies between 2 and 3, and therefore r may be made either equal to 2 or to 3: since then in this case $s=1$, we shall have $\frac{r+\frac{n}{r}}{2s}=\frac{5}{4}$ or $\frac{5}{3}$, according as r is taken equal to 2 or 3; therefore t must be the simplest fraction between $\frac{5}{4}$ and $\frac{5}{3}$, that is $\frac{3}{2}$; therefore t and consequently $ts=\frac{3}{2}$; whence $\sqrt{t^2s^2-n}=\frac{\sqrt{5}}{2}$; and so the binomial to be tried is $\frac{3-\sqrt{5}}{2}$, which succeeds; for the cube of the numerator is $72-32\sqrt{5}$, and the cube of the denominator is 8, and the former divided by the latter gives $9-4\sqrt{5}=9-\sqrt{80}$.

In all these cases where the value of r *cannot be expressed by the nearest whole number but very grossly, it will be more adviseable to express it by some simple fraction, which may easily be obtained thus: let the cube root of any given number be required, and let* a^3 *be the nearest less cube: now if to this cube* a^3 *be added* $aa+\frac{1}{3}a$, *you will have the cube of* $a+\frac{1}{3}$ *nearly; and if again to this last sum be added* $aa+a$, *you will have the cube of* $a+\frac{2}{3}$ *nearly: whence it will be very easy to see whether the cube root sought lies between* a *and* $a+\frac{1}{3}$, *or between* $a+\frac{1}{3}$ *and* $a+\frac{2}{3}$, *or between* $a+\frac{2}{3}$ *and* $a+1$. As for instance, in this last example we had occasion for the cube root of 18, where the nearest less cube was 8, and $a=2$; now if to this cube 8 be added $a^2+\frac{1}{3}a$, that is $4\frac{2}{3}$, you will have $12\frac{2}{3}$ for the cube of $2\frac{1}{3}$; add again $aa+a$ or 6, and you will have $18\frac{2}{3}$ for the cube of $2\frac{2}{3}$; whence I infer that the cube root of 18 lies between $2\frac{1}{3}$ and $2\frac{2}{3}$, but much nearer $2\frac{2}{3}$: make then $r=\frac{8}{3}$, and since $n=1$ and $s=1$, you will have $\frac{r+\frac{n}{r}}{2s}=\frac{3}{2}+\frac{1}{48}$; therefore t and consequently $ts=\frac{3}{2}$ as before.

EXAMPLE 4.

Let the cube root of $\frac{13}{9}\sqrt{12}+5$ be required. Here $aa=\frac{169}{81}\times 12$ $\frac{676}{27}$, $bb=25=\frac{675}{27}$, $a^2-b^2=\frac{1}{27}$, $n=\frac{1}{3}$, $z=1$. Again, $\frac{676}{27}$ $=25+$, whose square root is 5; therefore $\overline{a+b}\times z=10$, whose cube root is 2; therefore $r=2$, and $r+\frac{n}{r}=2\frac{1}{6}=2\frac{2}{10}-$. Again, $\frac{676}{27}$ $=26\times\frac{26}{27}=\overline{26\times\frac{26}{81}\times 3}$; therefore $\sqrt{\frac{676}{27}}$ or $az=\frac{26}{9}\sqrt{3}$; therefore $s=\sqrt{3}=1\frac{7}{10}$; therefore $\frac{r+\frac{n}{r}}{2s}=\frac{2\frac{2}{10}}{3\frac{4}{10}}$: now $\frac{2\frac{2}{10}}{3\frac{3}{10}}=\frac{2}{3}$; therefore $\frac{2\frac{2}{10}}{3\frac{4}{10}}$ $=\frac{2}{3}$ nearly; so I make $t=\frac{2}{3}$; whence $ts=\frac{2\sqrt{3}}{3}=\frac{\sqrt{12}}{3}$, and t^2s^2 $-n=\frac{12}{9}-\frac{1}{3}=1$; therefore $\sqrt{t^2s^2-n}=1$, and the root to be tried is $1+\frac{\sqrt{12}}{3}=\frac{3+\sqrt{12}}{3}$, which succeeds; for if the cube of the numerator be divided by the cube of the denominator, the quotient will be $\frac{13}{9}\sqrt{12}+5$.

In this example I found the irrational part first, because of some seeming difficulties in it; but the rational part is more easily found: for since $r=2$, $n=\frac{1}{3}$, and in this case, $s=1$, we shall have $\frac{r-\frac{n}{r}}{2s}=1-\frac{1}{12}$; therefore ts the rational part of the root sought equals 1, and $\sqrt{t^2s^2+n}$ the other part equals $\frac{2}{\sqrt{3}}=\frac{2\sqrt{3}}{3}=\frac{\sqrt{12}}{3}$ as before.

EXAMPLE 5.

Let the cube root of $68+\sqrt{4374}$ be required. Here $a^2=4624$, $b^2=4374$, $a^2-b^2=250=2\times5\times5\times5$; therefore in this case a^2-b^2 must be multiplied into 4 to make it a cube, to wit $8\times5\times5\times5$, whose side is $2\times5=10$; therefore $n=10$, $zz=4$, $z=2$, and $\sqrt[3]{z}=\sqrt[3]{2}$. Again, $\sqrt{4374}=66$; therefore $\overline{a+b}\times z=\overline{68+66}\times 2=268$, whose cube root

root is almoſt in the middle between $6\frac{1}{3}$ and $6\frac{2}{3}$; ſo I make $r=6\frac{1}{2}$; whence $r+\frac{n}{r}=6\frac{1}{2}+\frac{20}{13}=8\frac{1}{26}$; but $s=1$; therefore $\frac{r+\frac{n}{r}}{2s}=4\frac{1}{52}$: make ts the rational part, equal to 4, and you will have $ttss-n=6$; whence $\frac{ts+\sqrt{ttss-n}}{\sqrt[s]{z}}$ the root to be tried, is $\frac{4+\sqrt{6}}{\sqrt[3]{2}}$, and it ſucceeds; for if $136+54\sqrt{6}$ the cube of the numerator, be divided by 2 the cube of the denominator, the quotient will be $68+27\sqrt{6}=68+\sqrt{4374}$.

EXAMPLE 6.

Let the fifth root of $29\sqrt{6}+41\sqrt{3}$ be required. Here $a^2=5046$, $b^2=5043$; therefore $a^2-b^2=3$; therefore in this caſe a^2-b^2 muſt be multiplied by 81, to have a rational fifth root; for then it will be 243, whoſe fifth root is 3; therefore $n=3$, and $z=9$. Again, $\sqrt{5046}=71$, and $\sqrt{5043}=71$; therefore $\overline{a+b}\times z=142\times 9=1278$, whoſe fifth root is $4+$; therefore $r=4$, and $r+\frac{n}{r}=4\frac{3}{4}$. Laſtly, $az=29\times\sqrt{6}\times 9$; therefore $s=\sqrt{6}=2\frac{1}{2}-$; therefore $\frac{r+\frac{n}{r}}{2s}=\frac{4\frac{3}{4}}{5}=1-\frac{1}{20}$; therefore $t=1$, $ts=\sqrt{6}$, $\sqrt{t^2s^2-n}=\sqrt{3}$, and $\frac{ts+\sqrt{t^2s^2-n}}{\sqrt[s]{z}}$ of the root to be tried, is $\frac{\sqrt{6}+\sqrt{3}}{\sqrt[5]{9}}$, and it ſucceeds; for according to *Newton*'s theorem, the fifth power of $\sqrt{6}+\sqrt{3}$ is $36\sqrt{6}+180\sqrt{3}+180\sqrt{6}+180\sqrt{3}+45\sqrt{6}+9\sqrt{3}=261\sqrt{6}+369\sqrt{3}$; which being divided by 9, the fifth power of the denominator, gives $29\sqrt{6}+41\sqrt{3}$ the number firſt propoſed.

THE ELEMENTS of ALGEBRA

BOOK X. IN TWO PARTS.

I. Of Equations in general, and their roots.

II. Of cubic and biquadratic equations in particular.

PART I.

Of equations in general, and their roots.

DEFINITIONS.

427. EQUATIONS *are usually denominated from the highest power of the unknown quantity that is alone concerned in them.* Thus if the unknown quantity arises to the third power, the equation is called a cubic equation; if to the fourth, a biquadratic; if to the fifth, a quadrato-cubic; if to the sixth, a bicubic, &c. *But if an equation can (by a bare reduction of the powers of the unknown quantity) be brought down to a lower form, it then receives it's denomination from that form.* Thus the equation $x^6-ax^3+b=0$, where x is the unknown quantity, is called a quadratic, because, by substituting y for x^3, it may be reduced to one: thus again, the equation $x^6-ax^4+bxx-c=0$ is called a cubic equation, because it may be reduced to one by substituting y for xx.

The root of an equation is such a quantity, as being substituted for the unknown quantity, will make the two sides equal one to the other; or (which is the same thing) if all the parts of an equation be thrown to one side, and so be made equal to nothing on the other, then that is said to be the root of the equation, which being substituted for the unknown quantity, will make the whole equation to vanish. Thus the roots of the equation $xx-8x+15=0$ are 3 and 5, because either of these numbers being put for x will make

make the whole equation to vanish; and there are no others besides in the whole scale of numbers that will have that effect.

428. *Every equation hath as many roots, possible and impossible, as there are dimensions in the highest power of the unknown quantity.* Thus every cubic equation has three roots, every biquadratic four, &c. But it often happens that some of these roots are impossible, and sometimes that they are all so, if the index of the highest power of the unknown quantity be an even number; for if it be otherwise, the equation will always have one root at least possible, as will be evident from what will be said hereafter when we come to treat of the constitution of equations. Thus the equation $x^3=1$ hath indeed three roots, but it hath but one real root, to wit unity, the other two being impossible, to wit, $\frac{-1+\sqrt{-3}}{2}$ and $\frac{-1-\sqrt{-3}}{2}$. Let us see however how these impossible numbers being substituted for x, will make $x^3=1$: and first let us make $x=\frac{-1+\sqrt{-3}}{2}$. Make $-1=a$, and $\sqrt{-3}=b$; then you will have $x^3=\frac{a^3+3a^2b+3ab^2+b^3}{8}$, whereof $a^3=-1$; $3aab=+3\times\sqrt{-3}=3\sqrt{-3}$: again, as $b=\sqrt{-3}$, we shall have $b^2=-3$, and $3ab^2=-3\times-3=+9$: lastly b^3, or $b^2\times b=-3\sqrt{-3}$; therefore $x^3=\frac{-1+3\sqrt{-3}+9-3\sqrt{-3}}{8}=\frac{-1+9}{8}=1$. Let us now put $\frac{-1-\sqrt{-3}}{2}$ for x; and making $-1=a$, and $-\sqrt{-3}=b$, we shall have the first term of x^3, wherein b is not concerned, and the third term, wherein b^2 is only concerned, the same as before; but the second and fourth terms, wherein b and b^3 are concerned, will have their signs changed: therefore in this case $x^3=\frac{-1-3\sqrt{-3}+9+3\sqrt{-3}}{8}=1$.

Of the rise of these impossible roots in quadratic equations we have spoken already; and how by their means they creep into equations of higher degrees, shall be shewn hereafter.

Of the generation of equations.

To form an equation that shall have any number of given roots.

—). *This is easily done, first by changing the signs of all the roots given, and then joining them severally to some indeterminate quantity; for if the quantities*

quantities thus constituted be multiplied all together, and the product be made equal to nothing, you will have the equation desired. As for instance, let it be required to form an equation that shall have these three roots and no more, to wit 1, 2 and 3: now if the signs of these roots be changed, and they be severally joined with the indeterminate quantity x, they will constitute the quantities $x-1$, $x-2$, $x-3$; multiply all these together, and making their product equal to nothing, you will have the equation desired, to wit, $x^3-6xx+11x-6=0$, whose roots are the numbers first proposed, to wit 1, 2 and 3.

This may easily be tried by substituting these numbers (one after another) for x in the equation: but the reason of this method will best be perceived by resolving again the equation into it's first factors thus, $\overline{x-1}\times\overline{x-2}\times\overline{x-3}=0$; I say then that the roots of this equation are these three and no more, to wit, 1, 2 and 3. For first, if 1 be substituted for x in the equation $\overline{x-1}\times\overline{x-2}\times\overline{x-3}=0$, you will have $\overline{1-1}\times\overline{1-2}\times\overline{1-3}=0$; which must necessarily be so, because one of the factors, to wit $1-1$, equals 0. Make now x equal to 2 in the foregoing equation, and you will have $\overline{2-1}\times\overline{2-2}\times\overline{2-3}=0$, because $2-2$ is so. Lastly, make x equal to 3, and you will have $\overline{3-1}\times\overline{3-2}\times\overline{3-3}=0$, because $3-3$ is so: therefore the three roots of this equation are the numbers 1, 2 and 3, from the very definition of a root.

Let us now see whether this equation can have any other roots besides those already mentioned; and (if possible) let r be a fourth root different from any of these: then if r be substituted for x in the equation, we should have $\overline{r-1}\times\overline{r-2}\times\overline{r-3}=0$; but now as r is different from any of the numbers 1, 2 and 3, by the supposition, it follows that every one of the factors $r-1$, $r-2$, $r-3$ must be something; and if so, it will be impossible for these, when multiplied together, to produce nothing; therefore the foregoing equation admits of no other roots but those already mentioned.

If I would add a fourth root to the foregoing equation, to wit -5, I might either put it thus, $\overline{x-1}\times\overline{x-2}\times\overline{x-3}\times\overline{x+5}=0$, or I might multiply the cubic equation before found, to wit, $x^3-6xx+11x-6$ by $x+5$, and so make the product $x^4-x^3-19xx+49x-30=0$.

Whenever an equation may be wholly divided by the simple power, or by the square, or the cube of the unknown quantity without a remainder, it is an infallible argument that one or two or three of the roots of such an equation are equal to nothing. Thus the equation $x^4-6x^3+11xx-6x=0$ may be divided by x without a remainder; whence I conclude that one of it's roots is equal to nothing, as will be evident, either by substi-

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tuting nothing for x in the equation, or by confidering the equation as refolved into it's conftituent factors thus, $x \times \overline{x-1} \times \overline{x-2} \times \overline{x-3} = 0$; as if I had faid $\overline{x-0} \times \overline{x-1} \times \overline{x-2} \times \overline{x-3} = 0$: whence it is evident that 0 has as good a title to be a root of this equation as any of the reft.

430. *From what has been laid down in the laft article it appears, that if any number of equations whofe parts on one fide are all equal to nothing on the other, be multiplied together, they will produce an equation of a fuperior form, whofe roots will be the fame with their's.* Thus if the following equations be multiplied together, *viz.* $x-1=0$, whofe root is 1, $x-2=0$, whofe root is 2, and $x-3=0$, whofe root is 3, they will produce the equation $\overline{x-1} \times \overline{x-2} \times \overline{x-3} = 0$, whofe roots are 1, 2 and 3, by the laft article. Again, let thefe equations be multiplied together, *viz.* $2x-3=0$, whofe root is $\frac{3}{2}$, $4x-5=0$, whofe root is $\frac{5}{4}$, and $6x-7=0$, whofe root is $\frac{7}{6}$, and they will produce the equation $\overline{2x-3} \times \overline{4x-5} \times \overline{6x-7} = 0$, whofe roots are $\frac{3}{2}$, $\frac{5}{4}$ and $\frac{7}{6}$: for if $\overline{2x-3} \times \overline{4x-5} \times \overline{6x-7} = 0$, we fhall have by divifion $\overline{x-\frac{3}{2}} \times \overline{x-\frac{5}{4}} \times \overline{x-\frac{7}{6}} = 0$, and the roots of this equation are $\frac{3}{2}$, $\frac{5}{4}$ and $\frac{7}{6}$ by the laft article. Laftly, let thefe two equations be multiplied together, to wit $xx-ax+b=0$, whofe roots we will call p and q, and $xx-cx+d=0$, whofe roots we will fuppofe to be r and s: I fay then that the roots of the equation $\overline{xx-ax+b} \times \overline{xx-cx+d} = 0$ will be p, q, r, s. For firft, as p is one of the roots of the equation $xx-ax+b=0$, it follows from the nature of a root, that if x be put equal to p, the quantity $xx-ax+b$ muft vanifh; but if $xx-ax+b=0$, then $\overline{xx-ax+b} \times \overline{xx-cx+d}$ muft be equal to nothing; therefore p is one of the roots of the equation $\overline{xx-ax+b} \times \overline{xx-cx+d} = 0$: and the fame may be obferved of all the other roots q, r, s.

Hence it is that impoffible roots creep into equations of all orders and degrees whatever: for if the roots of one or more quadratic equations that enter into the compofition of an equation of a fuperior form, be impoffible, the equation fo formed muft neceffarily have the fame impoffible roots. And this is the reafon why *In every equation, the number of impoffible roots is always even*; becaufe the roots of a quadratic equation muft always be both poffible, or both impoffible, as was fhewn in another place: *but if the index of the higheft term of an equation be an odd number, it muft have at leaft one root poffible,* becaufe fuch an equation cannot be formed wholly of quadratics; it muft have a fimple equation in it's compofition, over and above; and the root of a fimple equation is always poffible.

431. The converfe of the laft article will alfo be true, if well underftood; to wit, that *If any equation of a fuperior form be propofed, whofe parts on one fide are all equal to nothing on the other, and if the quantity which*

which in the equation is supposed equal to nothing, can be resolved into more simple factors, in all which the unknown quantity is more or less concerned, and lastly, if these factors be made each equal to nothing, you will then have a set of equations of an inferior rank, which all together will have the same roots with those of the equation proposed: but these roots must now be more easily obtained, as being to be extracted out of equations of a simpler kind.

It may perhaps be objected, that it is no good consequence to infer, that because a quantity is equal to nothing, therefore all it's constituent factors must be so too; since if but any one of these be equal to nothing, that will be sufficient to destroy the whole: but whoever makes this objection, does not apprehend my conclusion; my inference was not from a quantity's being equal to nothing, that therefore all it's constituent factors must be equal to nothing, or to any thing; that matter is purely arbitrary, and certainly the Analyst is at liberty to equate them to what he pleases, being under no restraint from the nature of the thing; but what I contend for is, that if he will have a set of equations, which all together will have the same roots as the superior equation proposed, he must then make all his factors equal to nothing.

When the unknown quantity signifies the same thing in different equations, it will then be lawful to multiply these equations together, and we may safely pronounce the product of all the antecedents on one side equal to the product of all the consequents on the other, that is, we may do this without changing the signification of the unknown quantity. But if in different equations the unknown quantity has different values, and these equations be multiplied together, there will then be but one case wherein we may conclude the product on one side equal to the product on the other, without taking the unknown quantity in a different sense in the conclusion from what it had in any of the premisses; and if the inference will not hold good in composition, much less will it in the resolution. But one case there is wherein equations may be thus united without changing the sense of the unknown quantity; and that is, when the second side of every constituent equation is nothing: this, I think, was thoroughly made out in the last article; and if the composition be just, we must allow of the resolution, or be liable to infinite contradictions and absurdities.

432. *Hence it is that the invention of divisors comes to be of any use in the resolution of equations*: for by finding all the divisors that will divide the quantity which in the equation is supposed equal to nothing, without any remainder, we can resolve it into it's most simple factors. As for instance, let this equation be proposed, $x^3=1$, or $x^3-1=0$: now if we examine the quantity x^3-1 according to art. 409, we shall find it hath this divisor of one dimension, to wit $x-1$, and that the quotient

is

is $xx+x+1$: make therefore $x-1=0$, and $xx+x+1=0$, and the equation propoſed will be reſolved into a ſimple equation and a quadratic: the root of the ſimple equation $x-1=0$ is 1, but the roots of the quadratic equation $xx+x+1=0$ are impoſſible roots, to wit, $\frac{-1+\sqrt{-3}}{2}$ and $\frac{-1-\sqrt{-3}}{2}$; but that either of theſe two laſt quantities, as well as the firſt, being ſubſtituted for x, will make $x^3=1$, was ſhewn in art. 428.

Again, let the equation be $x^3-xx-10x+6=0$. This quantity $x^3-xx-10x+6$ was examined in art. 409, and was found to have a diviſor, as $x+3$, and the quotient was $xx-4x+2$: make $x+3=0$, and $xx-4x+2=0$, and the root of the former equation will be -3, and the two roots of the latter $2+\sqrt{2}$ and $2-\sqrt{2}$; therefore the roots of the cubic equation propoſed will be -3, $2+\sqrt{2}$ and $2-\sqrt{2}$.

Laſtly, let the equation propoſed be $x^4-x^3-5xx+12x-6=0$. This quantity $x^4-x^3-5xx+12x-6$ was examined in art. 411, and was found to have this diviſor, to wit $xx+2x-2$, and the quotient was $xx-3x+3$: make $xx+2x-2=0$, and alſo $xx-3x+3=0$, and the biquadratic equation propoſed will now be reſolved into two quadratics; the roots of the former equation will be $-1+\sqrt{3}$ and $-1-\sqrt{3}$; but the roots of the latter equation will be impoſſible, to wit, $\frac{3+\sqrt{-3}}{2}$ and $\frac{3-\sqrt{-3}}{2}$; therefore the four roots of the biquadratic equation propoſed will be $-1+\sqrt{3}$, $-1-\sqrt{3}$, $\frac{3+\sqrt{-3}}{2}$, and $\frac{3-\sqrt{-3}}{2}$.

Of the coefficients of the terms of equations.

433. *If the roots of an equation have all their ſigns changed, and the equation itſelf be ſo ordered as that all it's terms on one ſide may be equal to nothing on the other, having unity for the coefficient of it's firſt term; I ſay then that the coefficient of the ſecond term will be the ſum of all the roots thus changed, the coefficient of the third term will be the ſum of all the different products that can be made out of them by multiplying them two and two together, the coefficient of the fourth term will be the ſum of all the different products that can be obtained from them by multiplying them three and three together, the coefficient of the fifth term will be the ſum of all the different products that can ariſe by multiplying them four and four together,* &c: *and thus may an equation be formed that ſhall have any number of given*

 roots,

roots, without continual multiplication. As for instance, suppose I would form an equation that shall have these four roots, 1, 2, 3 and -5: now these roots when their signs are changed, will be -1, -2, -3, $+5$, whose sum is -1; therefore -1 will be the coefficient of the second term. Again, all the different products that can be made out of them by multiplying them two and two together, are -1×-2, -1×-3, $-1\times+5$, -2×-3, $-2\times+5$, $-3\times+5$, $=+2, +3, -5, +6, -10, -15, =-19$; therefore the coefficient of the third term will be -19. All the different products that can arise by multiplying them three and three together, are $-1\times-2\times-3$, $-1\times-2\times+5$, $-1\times-3\times+5$, $-2\times-3\times+5$, $=-6, +10, +15, +30, =49$; therefore 49 will be the coefficient of the fourth term. Lastly, there will be but one product obtainable by multiplying them all four together, that is $-1\times-2\times-3\times+5=-30$; therefore -30 will be the last term of the equation, and the whole equation will now be formed, to wit, $x^4-x^3-19xx+49x-30=0$.

The truth of this proposition will easily appear by representing the roots in general terms thus: multiply $x-a$, $x-b$ and $x-c$ together, making the product equal to nothing, and you will have an equation whose roots are a, b and c, by art. 429; and this equation will be found to be $x^3 \overline{-a-b-c}\times xx \overline{+ab+ac+bc}\times x-abc=0$, which is usually written thus:

$$x^3 \begin{array}{l} -a \\ -b \\ -c \end{array} xx \begin{array}{l} +ab \\ +ac \\ +bc \end{array} x \; -abc=0.$$

If $x-a$, $x-b$, $x-c$ and $x-d$ be multiplied together, you will have an equation whose roots are a, b, c and d, and the equation will be found to be

$$x^4 \begin{array}{l} -a \\ -b \\ -c \\ -d \end{array} x^3 \begin{array}{l} +ab \\ +ac \\ +ad \\ +bc \\ +bd \\ +cd \end{array} xx \begin{array}{l} -abc \\ -abd \\ -acd \\ -bcd \end{array} x \; +abcd=0.$$

After the same manner may all the other cases be demonstrated, be their signs or their roots what they will.

From what has been said it follows, that if the coefficient of the first term of an equation be 1, *the last term will be the product of all the roots multiplied together, and consequently that no rational number can be the root of such an equation, that is not an affirmative or a negative divisor of the last term.*

Of

Of the number of affirmative and negative roots in an equation.

434. *When the roots of an equation are all possible, and all it's parts are thrown to one side so as to be made equal to nothing on the other, the number of affirmative and negative roots may be thus determined: as often as the signs change in passing from the highest term to the lowest, just so many roots of the equation will be affirmative; and as often as like signs follow one another, so many will be negative.* As for instance, in the equation $x^4 - x^3 - 19xx + 49x - 30 = 0$, where the order of the signs is $+$, $-$, $-$, $+$, $-$; $+$ in the first term being followed by $-$ in the second, argues one affirmative root; $-$ in the second term followed by $-$ in the third, argues one root to be negative; $-$ in the third term followed by $+$ in the fourth, discovers another affirmative root; and $+$ in the fourth term followed by $-$ in the fifth, a third: so that if all the roots of this equation be possible, there will be three affirmative roots, and one negative, which is true; for the roots of this equation are $+1$, $+2$, $+3$ and -5, as was shewn in the last article.

This rule is usually ascribed to our Countreyman *Harriott*, who was undoubtedly the first discoverer of most of those general properties of equations hitherto delivered, or to be delivered. But whosoever it was that first hit upon it, this is certain, that he left no demonstration of it; nor have I ever met with one in any treatise of Algebra that has hitherto fallen into my hands, though most of them make mention of the rule. And indeed, whoever considers the immense number of cases that must necessarily come under consideration in a demonstration of this nature, will not be very ready to attempt it universally; I say universally, for equations of lower degrees are less embarrassed, as I have shewn already in the case of quadratic equations, and shall further shew when I come more distinctly to consider cubics. On the other hand, it seems much more probable, that this rule was found out by experience than from any regular investigation of it: for let all the roots of an equation be affirmative, as $+a$, $+b$, $+c$; then the equation will be $\overline{x-a}\times\overline{x-b}\times\overline{x-c} = 0$, that is,

$$x^3 \begin{matrix} -a \\ -b \\ -c \end{matrix} xx \begin{matrix} +ab \\ +ac \\ +bc \end{matrix} x - abc = 0,$$

where there are as many changes of signs as there are roots. Let us now suppose all the roots of an equation to be negative, as $-a$, $-b$, $-c$; then the equation will be $\overline{x+a}\times\overline{x+b}\times\overline{x+c} = 0$, that is,

$$x^3 \begin{matrix}+a\\+b\\+c\end{matrix} xx \quad \begin{matrix}+ab\\+ac\\+bc\end{matrix} x \quad +abc=0,$$

where there are no changes of ſigns at all: and from the formation of equations in the laſt article it is very viſible, that theſe two extreme caſes will be the ſame, how high ſoever the equation riſes; that is, when all the roots are affirmative, there will be as many changes of ſigns as there are roots; and when none are affirmative, there will be no changes at all. Now the queſtion is, whether an ingenious artiſt would not take the hint, which could not well eſcape his obſervation, and immediately ſet himſelf to try, whether in all other caſes the number of affirmative roots be not equal to the number of variations in the ſigns as they follow one another.

This rule (as I before obſerved) *only takes place where all the roots of an equation are poſſible*; for if any of them happen to be otherwiſe, it will then be impoſſible to determine the ſigns of the real roots. Impoſſible quantities, properly ſpeaking, belong to no claſs, either of affirmatives or negatives, and yet they always appear under one form or the other; nay ſuch is the capriciouſneſs of theſe quantities, (if I may call them ſo,) that the very ſelf-ſame roots often appear in both ſhapes: of this I have given ſome inſtances in art. 109 with relation to quadratics; and give me leave to produce one or two more in this place.

The equation $x^3+qx-r=0$ is defective for want of a ſecond term, and therefore cannot be examined by the foregoing rule till that defect is ſupplied by $+0xx$ or $-0xx$: let us then put the equation $x^3+0xx+qx-r=0$, and then by the foregoing rule it appears, that this equation has one affirmative and two negative roots: let us now put the equation $x^3-0xx+qx-r=0$, which differs nothing from the former but in the manner of conceiving it; and now according to the foregoing rule, all the roots are affirmative. This is a plain indication that there lie concealed in this equation two impoſſible roots, aſſuming the ſhape of affirmative quantities in one light, and that of negatives in another.

Again, let this equation $x^3+pxx+3ppx-q=0$ be examined by the foregoing rule, and it will be found to have one affirmative, and two negative roots: let us now add another affirmative root, as $+2p$, to this equation, multiplying it by the equation $x-2p=0$, and we ſhall have

$$x^4 \quad -px^3 \quad +ppxx \quad \begin{matrix}-6p^3\\-q\end{matrix} x \quad +2pq=0:$$

this equation ought to have two affirmative and two negative roots; but according to the foregoing rule, all it's four roots are affirmative; which ſhews, that in the foregoing equation there were concealed two negative impoſſible roots, which are changed into affirmative ones in this.

435. But here the reader ought to be put in mind, that *It very often happens, especially in geometrical problems, that the roots of equations are possible, and yet the schemes to which they relate may exhibit them impossible, upon the account of some limitation or other in the problem which does not enter the equation.* An instance or two will clear up this matter. (See Plate vii. Fig. 61.)

Let ABC be a circle whose diameter is AC, and let AB be any chord inscribed in it; from B, the end of the chord, draw the line BD perpendicular to the diameter AC; and let it be required, having given the lines AB and AC, to compute AD, the segment of the diameter intercepted between the point A and D the foot of the perpendicular. Join BC, and call AD x: then will the similar triangles ADB and ABC give the following proportion, to wit, AC is to AB as AB is to AD or x; whence $x = \frac{AB^2}{AC}$; therefore x, the root of this equation, will be equally possible, whether AB be greater or less than AC; but the problem will not be possible unless the chord AB be less than the diameter AC. Here then we have an instance of a problem's producing an equation whose root continues to be possible, even when the problem ceases to be so, on account of the limitation abovementioned, to wit, that in the problem, AB must be less than AC. To explain this mystery, let AC and AB be any two given lines, and let it be required to assign a third proportional to them, which we will call x: since then AC is to AB as AB is to x, we have again $x = \frac{AB^2}{AC}$: but this problem is as unlimited as the equation it produces; for it is certain the two quantities AC and AB will admit of a third proportional, whether AB be greater or less than AC. The case then stands thus: here is an equation arising from a limited problem; but this equation is also intended to solve another problem that is absolutely unlimited; therefore it ought not to be expected that the equation should be liable to any restriction, whatever may be the case of the problem that produced it.

In the 122d and 123d articles we had an instance of two problems producing one and the same quadratic equation; and accordingly two roots were found which would equally solve the equation; but the problems themselves were under such different limitations, that the same root would not solve both problems, but one root solved one problem and the other root the other. And I believe I may venture to lay it down for a general observation, that whenever a problem produces an equation that admits of two or more roots, whereof there is but one that will solve the problem, I believe I say, we may generally conclude that there are other problems

problems producing the same equation, and that the rest of the roots are intended to solve those other problems.

Another instance to shew that the roots of equations may sometimes be possible when the nature of the problem to which they are applied may be so limited as absolutely to exclude them, take as follows.

Let it be required, having given the side and solid content of a cone, to find it's altitude, and the semidiameter of the base.

Let p be the semiperiphery of a circle whose semidiameter is 1; then will p be also the area of that circle, and the square of the semidiameter of every circle will be to it's area as 1 to p; which is as much as to say, that if the square of the semidiameter of any circle be multiplied by p, the product will be the area of the circle. This allowed, let s be the given side of the cone; and dividing the given solid content by p, call the quotient q, and pq will be the solid content itself. Put x for the altitude of the cone, and y for the semidiameter of the base; then will pyy be the area of the base, and $\frac{pxyy}{3}$ the solid content of the cone; therefore $\frac{pxyy}{3} = pq$, and $\frac{xyy}{3} = q$, and $yy = \frac{3q}{x}$. Again, $xx + yy = ss$, by the 47th of the first book of the Elements; (for by the side of the cone we mean the hypotenuse of the generating triangle, or a line drawn upon the surface of the cone from it's vertex to the circumference of the base;) therefore $yy = ss - xx = \frac{3q}{x}$, and $ssx - x^3 = 3q$, and $x^3 - ssx + 3q = 0$: this is a cubic equation, and when s and q are expounded in numbers, may easily be resolved, either by the rule of divisors already explained, or else by some of the rules hereafter to be delivered when we come to treat more particularly concerning the resolution of cubic equations. As for instance, let s the given side of the cone be equal to 5, and pq the given solid content be equal to $12p$; then will $q = 12$, and the equation $x^3 - ssx + 3q = 0$ will now be changed into this, $x^3 - 25x + 36 = 0$: now by the rule of divisors it appears, that the quantity $x^3 - 25x + 36 = 0$ admits of $x - 4$ for a divisor, without any remainder; therefore 4 is one of the roots of this equation. Divide now the equation $x^3 - 25x + 36 = 0$ by $\overline{x - 4}$, and you will have the equation $xx + 4x - 9 = 0$, which being resolved will give the other two roots, to wit, $-2 + \sqrt{13}$ and $-2 - \sqrt{13}$. This equation then, to wit $x^3 - 25x + 36 = 0$, has three real roots, whereof two are affirmative, as 4 and $-2 + \sqrt{13}$, and one is negative, as $-2 - \sqrt{13}$; but this negative root, though it will solve the equation, will by no means solve the problem; for it supposes an impossible case, to wit, that not only the altitude of the cone is negative, but greater than the side,

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as will be found upon tryal. But this is not all; for $12p$ the solid content of the cone, is an affirmative quantity; therefore if the altitude be negative, the area of the base, and consequently the square of the semidiameter of the base must be so too: this root then supposes what is not to be supposed, but is absolutely impossible in the nature of the thing. Hence it appears that this problem, though producing a cubic equation, admits however but of two solutions. If 5 be the side of the cone, and the altitude be taken either equal to 4, or to $-2+\sqrt{13}$, the solid content of the cone will be $12p$. The former case is easily tried, and the latter may be tried thus: make $rr=13$, and the altitude of the cone will be $r-2$, and the square of the altitude $rr-4r+4=17-4r$; subtract this from 25 the square of the side, and there will remain $4r+8$ for the square of the semidiameter of the base; therefore $\overline{4r+8}\times p$ will be the area of the base; multiply this into $\overline{r-2}$ the altitude, and you will have the product of the base into the altitude equal to $\overline{4rr+8r-8r-16}\times p=\overline{4rr-16}\times p=\overline{52-16}\times p=36p$, the third part whereof $12p$, is the solid content of the cone, as it ought; therefore the altitude $r-2$ was rightly assigned.

I observed before, that the negative root $-\sqrt{13}-2$ would solve the equation $x^3-25x+36=0$, though it would not solve the problem: for making $rr=13$ as before, and putting x for $-r-2$, we shall have $x^3=-r^3-6rr-12r-8$; put 13 for rr, and then you will have $x^3=-13r-78-12r-8=-25r-86$: again, $-25x=-25\times\overline{-r-2}=+25r+50$; therefore $x^3-25x=-25r-86+25r+50=-36$; add 36 to both sides, and you will have $x^3-25x+36=0$.

Of the transformation of equations.

436. In order to fit and prepare equations for a more easy resolution, it will be proper that the Analyst be well acquainted with all the various ways of transforming equations one into another, most whereof he will meet with in this and the following article: as

1st. *When there are fractions in any equation, they must be taken away after the same manner as in simple equations, to wit, by multiplying the whole equation into all the denominators successively: and if after this, the highest power of the unknown quantity is affected with any coefficient but unity, that coefficient must also be taken off in the following manner. Let the equation, when reduced to integral terms, be* $ax^3+bxx+cx-d=0$, *where the coefficient of the highest power of the unknown quantity is* a; *then assuming any indeterminate quantity, as* y, *make* $\frac{y}{a}=x$, *and substituting* $\frac{y}{a}$ *instead*

of

of x *in the equation, it will ſtand thus,* $\frac{y^3}{aa}+\frac{byy}{aa}+\frac{cy}{a}-d=0$; *multiply the whole equation by* aa, *and you will then have an equation wherein the coefficient of the higheſt power of the unknown quantity is unity, to wit,* $y^3+byy+acy-aad=0$; *and when by the reſolution of this equation the value of* y *is diſcovered, that of* $\frac{y}{a}$ *or* x, *the root of the primitive equation, will alſo be known.* Since then every equation may be reduced to another, wherein the coefficient of the higheſt power of the unknown quantity is unity, I ſhall hereafter conſider all equations in that form.

2dly. *After the ſame manner may ſurd quantities be ſometimes alſo thrown out of an equation.* As for inſtance, let the equation be $x^3 * +qx-\sqrt{r}=0$. Now to get rid of the ſurd quantity $\sqrt{r}$, ſubſtitute $\frac{y}{\sqrt{r}}$ inſtead of x in the equation, and it will ſtand thus, $\frac{y^3}{r\sqrt{r}} * +\frac{qy}{\sqrt{r}}-\sqrt{r}=0$; multiply the whole equation by $r\sqrt{r}$, and you will have $y^3 * +qry-rr=0$; and when in this equation the value of y is known, that of $\frac{y}{\sqrt{r}}$ or x will alſo be known.

3dly. *Thus alſo may the roots of any equation be multiplied or divided by any given number whatever.* As for inſtance, let the equation be $x^3+qx-r=0$, and let it be required to multiply the roots of this equation by 3; that is, let it be required to find an equation whoſe roots ſhall be triple of the roots of this, each of each; and it may be thus done: becauſe 3 is the multiplicator, make $3x=y$, and then ſubſtituting $\frac{y}{3}$ inſtead of x in the equation, you will have $\frac{y^3}{27}+\frac{qy}{3}-r=0$; multiply all by 27, and you will have $y^3+9qy-27r=0$. In like manner may the roots of an equation be divided by 3 when occaſion requires it, to wit, by making $\frac{x}{3}=y$, and ſo ſubſtituting $3y$ inſtead of x in the equation.

4thly. *If you would change all the affirmative roots of an equation into equal negatives, and* vice verſa, *it may thus be done: let the equation be* $x^4-x^3-19x^2+49x-30=0$, *whoſe roots are* $+1, +2, +3$ *and* -5; *then ſubſtituting* $-y$ *inſtead of* $+x$ *in the equation, you will have* $y^4+y^3-19yy-49y-30=0$, *an equation whoſe roots are* $-1, -2, -3, +5$.

5thly. *It will ſometimes alſo be of uſe to change all the roots of an equation into their reciprocals,* thus: reſume the foregoing equation $x^4-x^3-19x^2$

$+49x$

ones taken together. For since, by art. 433, the coefficient of the second term is equal to the sum of all the roots with their signs changed; when the second term is wanting, that is, when it's coefficient is equal to nothing, the sum of all the roots must be equal to nothing; that is, all the affirmative roots taken together must be equal and contrary to all the negative roots taken together. As for instance, the roots of the equation $y^3 * -49y+120 = 0$ will be found to be 3, 5 and -8; where the sum of the affirmative roots 3 and 5 is equal and contrary to the single root -8.

By this way of transformation, and by the resolution of a quadratic equation, may the third term of an equation be also taken away, which in some geometrical constructions of equations is required to be done. As for example, let the equation whose third term is to be taken away, be $x^4-3x^3+3x^2-5x-2=0$, and make $y-z=x$ as before; and then you will have $x^4=y^4-4zy^3+6z^2y^2$ &c, $-3x^3=-3y^3+9zy^2$ &c, $+3x^2=+3y^2$ &c; and now the equation $x^4-3x^3+3x^2$ &c $=0$ will be converted into this:

$$y^4 \begin{matrix}-4z\\-3\end{matrix}\, y^3 \begin{matrix}+6zz\\+9z\\+3\end{matrix}\, y^2 \text{ \&c} = 0,$$

whose third term is $\begin{matrix}+6zz\\+9z\\+3\end{matrix}\, y^2$: but this equation is to have no third term, by the supposition; therefore $6zz+9z+3=0$; divide by 6, and you will have $zz+\frac{1}{2}z+\frac{1}{2}=0$; transpose $\frac{1}{2}$, and compleat the square, and you will have $zz+\frac{1}{2}z+\frac{9}{16}=\frac{1}{16}$; whence by extracting the square root, $z+\frac{3}{4}=\pm\frac{1}{4}$, and $z=-\frac{1}{2}$ or -1, and $y-z=y+\frac{1}{2}$ or $y+1$; substitute therefore $y+\frac{1}{2}$ or $y+1$ for x, in the equation $x^4-3x^3+3x^2-5x-2=0$, and you will have either way an equation whose third term is wanting; for if $y+\frac{1}{2}$ be made equal to x, the equation will be $y^4-y^3 * -\frac{15}{4}y-\frac{65}{16}=0$: if $y+1$ be made equal to x, the equation will be $y^4+y^3 * -4y-6=0$.

Here follows a general theorem for taking off the third term.

Let the equation be $x^m+px^{m-1}+qx^{m-2}$ &c $=0$, *and make* $\frac{p^2}{m^2}-\frac{2q}{m\times m-1}=ss$, *and the quantity to be substituted for* x *will be* $y-\frac{p}{m}\pm s$, which the reader may trace out at his leisure. Here then we are to take notice, that *If* q *be affirmative, and greater than* $p^2\times\frac{m-1}{2m}$, *the quantity* s, *and consequently this extermination, will be impossible; in all other cases it will be possible.*

PART II.

Of cubic and biquadratic equations.

Of cubic equations.

438. *If all the roots of a cubic equation be real, that is, none of them impossible, and if all the parts of such an equation be placed on one side according to the dimensions of the unknown quantity, and so be supposed equal to nothing on the other side; I say then, that as often as unlike signs follow one another in passing from the first term to the last, so many roots of this equation will be affirmative, and as often as like signs follow one another, so many roots will be negative.*

CASE 1.

Let the three roots of a cubic equation be a, b and c, and all affirmative: then will the equation be

$$x^3 \begin{array}{l} -a \\ -b \\ -c \end{array} xx \begin{array}{l} +ab \\ +ac \\ +bc \end{array} x - abc = 0$$ by art. 433;

where the signs are $+$, $-$, $+$, $-$.

CASE 2.

Let now all the roots be negative, as $-a$, $-b$, $-c$: then the equation will be

$$x^3 \begin{array}{l} +a \\ +b \\ +c \end{array} xx \begin{array}{l} +ab \\ +ac \\ +bc \end{array} x + abc = 0,$$

where the signs are $+$, $+$, $+$, $+$; so that for three variations of signs in the last case, there are none in this.

CASE 3.

Let now the roots of the equation be $+a$, $+b$, $-c$, that is, let two of the roots be affirmative and one be negative; and the equation will be

$$x^3 \begin{array}{l} -a \\ -b \\ +c \end{array} xx \begin{array}{l} +ab \\ -ac \\ -bc \end{array} x + abc = 0;$$

therefore in this case the absolute or last term of the equation will always be affirmative. Let the quantity c when taken affirmatively be less than

$\frac{ab}{a+b}$,

$+49x-30=0$, whose roots are 1, 2, 3 and -5; and let it be required to find an equation whose roots shall be the reciprocals of these: substitute $\frac{1}{y}$ instead of x, and the equation will be $\frac{1}{y^4}-\frac{1}{y^3}-\frac{19}{y^2}+\frac{49}{y}-30=0$; multiply the whole equation by y^4, and you will have $1-y-19y^2+49y^3-30y^4=0$, or $30y^4-49y^3+19y^2+y-1=0$, an equation whose roots are $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}$ and $\frac{-1}{5}$. Where it may be observed, that -5, the greatest root before, that is, the most remote from nothing, is now $-\frac{1}{5}$, the nearest to nothing.

437. *There is another transformation of equations no less useful than the foregoing, especially in the resolution of cubic and biquadratic equations, which is that, whereby the second term of any equation may be taken away. As for example, let this general equation be proposed,* x^m+px^{m-1} &c $=0$, *(for we shall have no occasion for more terms,) and let it be required from this equation to derive another which shall have no second term, and whose roots shall have a known relation to the roots of this.* Assuming any two indeterminate quantities y and z, make $y-z=x$, and substitute $y-z$ instead of x in the equation, and you will have $x^m=y^m-mzy^{m-1}$ &c, by *Newton*'s theorem for the evolution of a binomial; you will have moreover $px^{m-1}=py^{m-1}$ &c, and the equation x^m+px^{m-1} &c $=0$ will now be changed into this, $y^m \begin{matrix}+p\\-mz\end{matrix} y^{m-1}$ &c $=0$, whose second term is $\begin{matrix}p\\-mz\end{matrix} y^{m-1}$: but this equation is designed to have no second term; therefore to remove this term out of the equation, we must suppose $p-mz=0$; in which case we shall have $z=\frac{p}{m}$, and $y-z$ or $x=y-\frac{p}{m}$.

The rule therefore for striking out the second term is this. *From* y, *the intended root of the new equation, subtract* $\frac{p}{m}$, *the coefficient of the second term of the given equation divided by the number of dimensions in the first, and you will have a residual, as* $y-\frac{p}{m}$, *which being substituted for* x, *the root of the equation proposed, will change it into a new one whose second term is wanting, and whose roots have all a known relation to the roots of the equation given: for if from the roots of the new equation be subtracted* $\frac{p}{m}$, *you will have the roots of the equation first proposed.* As for example, let it be required to take away the second term of this equation, $x^3-6xx-37x+210=0$: Here p, the coefficient of the second term, is -6, and this divided by 3,

 the

the number of dimensions in the first term, quotes —2, which being subtracted from y, gives $y+2$; therefore I substitute $y+2$ instead of x in the equation, and the work stands thus:

$$\begin{array}{r|r}
x^3 & y^3+6y^2+12y+8 \\
-6x^2 & -6y^2-24y-24 \\
-37x & -37y-74 \\
+210 & +210 \\
\hline
 & y^3 \;*\; -49y+120=0.
\end{array}$$

Therefore the equation $x^3-6xx-37x+210=0$ is now converted into this, $y^3 * -49y+120=0$, whose second term is wanting, and whose roots, when extracted by rules hereafter to be given, will be found to be 3, 5 and —8; therefore the roots of the former equation were 5, 7 and —6, to wit, greater by 2 than those of the latter, because x was made equal to $y+2$. *And thus may all the roots of an equation be increased or diminished by any known quantity whatever, even so far as to become all affirmative, or all negative.* As for instance, suppose that in the equation first proposed, instead of making $y+2=x$, I had made $y-7=x$; I should then have had an equation whose roots would have been 12, 14 and 1, to wit greater by 7 than those of the equation proposed, and all affirmative: on the other hand, if I had made $y+8=x$, I should have had an equation whose roots would have been —3, —1, —14, that is, less by 8 than those of the equation first proposed, and all negative.

When the coefficient of the second term of an equation cannot be divided by the number of dimensions in the first without a fraction, the best way will be to multiply the roots of the equation by the denominator of such a fraction, and then to take away the second term out of the equation resulting from this multiplication. Ex gr. let the equation be $x^3+2xx-3x-4=0$. Now because 2 the coefficient of the second term, cannot be divided by 3 the number of dimensions in the first, without a fraction, as $\frac{2}{3}$, multiply the roots of the equation by the denominator 3, by making $3x=y$, and so putting $\frac{y}{3}$ instead of x in the equation, and you will have $\frac{y^3}{27}+\frac{2yy}{9}-\frac{3y}{3}-4=0$; multiply all by 27, and you will have $y^3+6y^2-27y-108=0$: take now away the second term of this equation, by substituting $z-2$ instead of y, and you will have a third equation whose second term is wanting, and which, for that reason, will be the more easily resolved. When all the roots z of this last equation are discovered, make $z-2=y$, and $\frac{y}{3}=x$, and you will have all the roots of the equation first proposed.

When the second term of an equation is wanting, it is an argument that all the affirmative roots taken together are equal and contrary to all the negative ones

$\frac{ab}{a+b}$; then it will be much leſs than $a+b$, which is equal to $\frac{aa+2ab+bb}{a+b}$; therefore $-a-b+c$ the coefficient of the ſecond term of the equation will be negative: it is evident alſo, ſince $\frac{ab}{a+b}$ is greater than c, that ab will be greater than $ac+bc$, and conſequently that $\overline{ab-ac-bc}$ the coefficient of the third term will be affirmative; therefore if c be leſs than $\frac{ab}{a+b}$, the ſigns of the terms of the equation will be $+, -, +, +$. Let now the quantity c be greater than $\frac{ab}{a+b}$, but ſtill leſs than $a+b$; and by a like way of reaſoning as before, the ſecond term of the equation will be found to be negative, and the third to be negative alſo; therefore if c lies between $a+b$ and $\frac{ab}{a+b}$, the ſigns of the terms will be $+, -, -, +$. Laſtly, let c be greater than $a+b$; then will the ſecond term of the equation be affirmative, the third will be negative, and the ſigns will now be $+, +, -, +$.

CASE 4.

Let now the ſigns of theſe roots be all changed by changing the ſign of x in the equation, which indifferently repreſents them all: then it is plain, that the ſigns of the firſt and third terms of the equation will be by this means changed, but theſe of the ſecond and fourth terms will ſtill remain; and now we ſhall have two negative roots, and one affirmative; and where in the former caſe the ſigns were $+, -, +, +$, they will now be $-, -, -, +$; where in the former caſe the ſigns were $+, -, -, +$, they will now be $-, -, +, +$; and laſtly where in the former caſe the ſigns were $+, +, -, +$, they will now be $-, +, +, +$: therefore in paſſing from the firſt term to the laſt of a cubic equation, there will always be as many variations of ſigns as there are affirmative roots, and *vice verſa*.

439. *If the ſecond term of a cubic equation be taken away, and ſo the equation be reduced to this form,* $x^3 \pm px = \pm q$; *I ſay then, that whatever quantities be the roots of the equation* $x^3 \pm px = +q$, *the ſame with their ſigns changed will be the roots of the equation* $x^3 \pm px = -q$; *and* vice verſa. For firſt, let the equation be $x^3+px=+q$; then changing the ſign of x, and conſequently of all the roots repreſented by it, the equation will be $-x^3-px=+q$; but this equation is the ſame with $x^3+px=-q$, as ap-

appears by transposition; therefore the roots of the equation $x^3+px=-q$ are equal and contrary to the roots of the equation $x^3+px=+q$.

In like manner if the equation be $x^3-px=+q$, by changing the sign of x we shall have $-x^3+px=+q$, which is the same with $x^3-px=-q$: whence it follows, that the same operation which finds the roots of the equation $x^3\pm px=+q$, will find the roots of the equation $x^3\pm px=-q$, to wit, by changing the signs of the former roots.

440. *Every cubic equation may be reduced to this form, to wit,* $x^3\pm3aax=\pm2aab$: for having cleared the first term from it's coefficient by art. 436, and taken away the second by art. 437, the equation will be reduced to this form, $x^3\pm px=\pm q$: make $+\frac{p}{3}=aa$ and $\frac{q}{2aa}=b$, and then the equation will be $x^3\pm3aax=\pm2aab$.

441. *Let* x *and* y *be two variable quantities, but in a constant relation to each other; as for instance, let* y *be universally equal to* $x^3-3aax-2aab$, *let the value of* x *be what it will: then if we suppose* x *first of all to be an infinite affirmative quantity, and from that state to flow downwards through all degrees of affirmative and negative magnitude into an infinite negative, we may mark down in a table some of the principal stations of* x *during this deflux, and overagainst them the correspondent values of* y, thus:

When x is infinite, y will be so too: for if x be infinite, xx, and $xx-3aa$ will be infinite, and consequently $x\times xx-3aa$, that is, x^3-3aax, and $x^3-3aax-2aab$ or y will be infinite; therefore overagainst *infinite* in the column signed x denoting the first state of x, write *infinite* in the column signed y denoting the like state of y.

The next station of x to be taken notice of during this deflux, is when $x=2a$; in which case x^3 will be $8a^3$, and $-3aax$ will be $-6a^3$, and $x^3-3aax-2aab$ or y will be $2a^3-2aab$; therefore putting down $2a$ in the column signed x, overagainst it write $2a^3-2aab$ in the column signed y.

When $x=a\times\sqrt{3}$, we shall have $x^3=3a^3\times\sqrt{3}$, and $-3aax=-3a^3\times\sqrt{3}$, in which case $x^3-3aax-2aab$ or $y=-2aab$; therefore write $a\times\sqrt{3}$ in the column x, and overagainst it $-2aab$ in the column y.

When $x=a$, y will be $-2a^3-2aab$, both which put down.

When $x=0$, $y=-2aab$.

When $x=-a$, $y=+2a^3-2aab$.

When $x=-a\times\sqrt{3}$, $y=-2aab$.

When $x=-2a$, $y=-2a^3-2aab$.

When x is an infinite negative, y will also be an infinite negative: all which must be registered, and so the table will be finished, which the reader (if he pleases) may transcribe into a piece of paper and lay it before him while he reads the following article.

X

x	y
+ infinite	+ infinite
$+2a$	$+2a^3-2aab$
$+a\times\sqrt{3}$	$-2aab$
$+a$	$-2a^3-2aab$
0	$-2aab$
$-a$	$+2a^3-2aab$
$-a\times\sqrt{3}$	$-2aab$
$-2a$	$-2a^3-2aab$
— infinite	— infinite.

442. *Every cubic equation of this form*, $x^3-3aax=\pm 2aab$ *will have all it's roots possible, provided that* b *be not greater than* a *or less than* $-a$, *but lies between those two limits.* This will be sufficiently demonstrated if we prove it of the equation $x^3-3aax=+2aab$ or $x^3-3aax-2aab=0$, by art. 439. Let us then suppose x and y to be two variable quantities as in the last article, and let y be universally equal to $x^3-3aax-2aab$ as before; then to enquire into the roots of the equation $x^3-3aax-2aab=0$, will be the same thing as to enquire into the values of x when $y=0$. Now in the foregoing table when x was $2a$, y was $2a^3-2aab$, which is an affirmative quantity, because b is supposed less than a: when x was $a\times\sqrt{3}$, y was $-2aab$ a negative quantity; therefore while x flowed down through all degrees of magnitude from $2a$ to $a\times\sqrt{3}$, y flowed down from an affirmative to a negative state: but no diminishing quantity can pass through all degrees of magnitude below it, out of an affirmative into a negative state, but it must at least once pass through nothing: therefore of all the intermediate values of x between $2a$ and $a\times\sqrt{3}$, there must be one where y or $x^3-3aax-2aab$ must be equal to nothing; and that value will be a root of that equation.

When x was equal to nothing, y was equal to $-2a^2b$ a negative quantity; and when x was $-a$, y was $+2a^3-2a^2b$ an affirmative quantity; therefore while x flowed downwards from nothing to $-a$, y flowed upwards from a negative to an affirmative state, and again must have passed through nothing; therefore we have another root of this equation lying between 0 and $-a$.

When x was $-a$, y was $+2a^3-2a^2b$ an affirmative quantity; and when x was $-a\times\sqrt{3}$, y was $-2a^2b$ a negative quantity; here then we find a third root of the equation, lying between $-a$ and $-a\times\sqrt{3}$: so that upon the whole matter we have pointed at three roots of the equation proposed, and three are as many as it will admit of; the greatest whereof is affirmative, and lies between $2a$ and $a\times\sqrt{3}$, or at least does not transcend those.

those limits; and the other two are negative, one lying between o and $-a$, and the other between $-a$ and $-a\times\sqrt{3}$.

This is upon a supposition that the equation is $x^3-3a^2x=+2a^2b$, that is, that b is affirmative: but if b be negative, that is, if the equation be $x^3-3aax=-2aab$, all things will happen contrarywise by art. 439; that is, the greatest root will now be negative, lying between $-2a$ and $-a\times\sqrt{3}$, and the other two roots will be affirmative, one lying betwixt o and $+a$, and the other betwixt $+a$ and $+a\times\sqrt{3}$.

If $b=+a$, which is the highest limit prescribed it by the theorem, the equation will be $x^3-3aax=2a^3$, in which case we shall have $x^3-3aax-2aab=2a^3-2aab$: but when $x^3-3aax-2aab$ or $y=2a^3-2aab$, x will be $+2a$, or $-a$ by the table; therefore in this case two of the roots of the equation will be $+2a$ and $-a$; but if one of the negative roots be $-a$, the other must also be $-a$, because both together must be equal and contrary to the affirmative root $+2a$, to destroy the second term of the equation, by art. 437: therefore when b rises as high as a, one of the limits prescribed it by the theorem, the affirmative root will ascend to $2a$, which is as high as it can rise, and the two negative roots, whereof one lies between o and $-a$ and the other between $-a$ and $-a\times\sqrt{3}$, will now meet in their common limit $-a$.

If $b=-a$, which is as low as it can descend according to the theorem, all things will happen contrary to the former case; that is, the greatest root will now be negative, and equal to $-2a$, which is it's lowest limit; and the two affirmative roots lying between o and $+a$, and between $+a$ and $+a\times\sqrt{3}$, will in this case meet in their common limit $+a$.

Lastly if b, and consequently $2aab=0$, that is, if the equation be $x^3-3aax=0$, one of the roots x will be equal to nothing by art. 429: and if x^3-3aax be divided by x, and the quotient $xx-3aa$ be made equal to nothing, you will have the other two roots of the equation, to wit $a\times\sqrt{3}$ and $-a\times\sqrt{3}$; therefore if $b=0$, the three roots of the equation will be $a\times\sqrt{3}$, o, and $-a\times\sqrt{3}$; that is, the greatest root, which lies between $+2a$ and $+a\times\sqrt{3}$, will now sink into it's lowest limit $+a\times\sqrt{3}$; and the greater negative root, which lies between $-a$ and $-a\times\sqrt{3}$, will also sink into it's lowest limit $-a\times\sqrt{3}$; but the other negative root lying between o and $-a$, will in this case ascend into it's highest limit o.

443. *Setting aside the case of the last article, I say that in all other cases, the equation* $x^3\pm3aax=\pm2aab$ *can have only one root possible, which root will be affirmative or negative according as* b *is so, that is, according as* $+$ *or* $-2aab$ *is concerned in the equation.* To demonstrate which, I need only to consider two of these equations, to wit $x^3-3aax=+2aab$, where we supposed b greater than a, and $x^3+3aax=+2aab$, where b may be any affirmative quantity whatever.

CASE

Case I.

Let us then consider that equation first, which borders next upon the equation of the last article, to wit, $x^3-3a^2x=+2a^2b$, or $x^3-3a^2x-2a^2b=0$, where b is supposed greater than a. In art. 107 it was demonstrated, that if the two roots of a quadratic equation, by approaching towards each other, come at last to be equal, the next step will be into a state of impossibility: but if this be the case of a quadratic equation, it must be the case of any other equation whatever that is above a quadratic: for if r and s be two roots of any equation whatever, they will also be the two roots of the quadratic equation $\overline{x-r}\times\overline{x-s}=0$.

To apply this now to our present case; it appears from the last article, that in the equation $x^3-3a^2x=2a^2b$ where b is less than a, the nearer it approaches towards a, the nearer will the two negative roots approach towards one another; when b becomes equal to a, those two roots will be equal to one another; and therefore when b becomes greater than a, the two roots must be impossible. But because this indirect way of reasoning may probably not go down with all sorts of readers, I shall demonstrate the thing more directly and more distinctly thus:

Let x and y be variable quantities related to each other as in the two last articles, and from the table there referred to it appears, that when x is infinite, y will also be infinite; when $x=2a$, y must be $2a^3-2a^2b$, which (we are to take notice) is now a negative quantity, because b is supposed greater than a; therefore whilst x flows from an infinite to a finite state so as to be equal to $2a$, y flows from a like state of infinity into a finite negative, and therefore must have passed through nothing in the mean time; therefore the equation $x^3-3a^2x=2a^2b$ must have one real affirmative root, and greater than $2a$. Let us call this affirmative root r; then will $x-r$ be a divisor of the quantity $x^3-3a^2x-2a^2b$, by art. 431 and 432: let it then be divided by $x-r$, and the quotient will be $\overline{x^2+rx+r^2-3a^2}$, and the remainder $r^3-3a^2r-2a^2b$ will be nothing, because as r is a root of the equation $x^3-3a^2x-2a^2b=0$, if r be put instead of x in that equation, you will have $r^3-3a^2r-2a^2b=0$; therefore if $x^3-3a^2x-2a^2b$ be divided by $x-r$, the quotient will be $xx+rx+r^2-3a^2$, and there will be no remainder: make this quotient equal to o, and you will have a quadratic equation including the other two roots of the cubic equation proposed. Since then $xx+rx+r^2-3a^2=0$, you will have by transposition, $xx+rx=3a^2-r^2$; by compleating the square, $xx+rx+\frac{1}{4}rr=3a^2-\frac{3}{4}rr$; and by extracting the square root, $x+\frac{1}{2}r=\pm\sqrt{3a^2-\frac{3}{4}rr}$: but it has been proved already, that r is greater than $2a$; whence r^2 is greater than $4a^2$, and $\frac{1}{4}r^2$ greater than aa, and $\frac{3}{4}r^2$ greater than $3a^2$; there-

fore

fore $3a^2-\frac{1}{4}r^2$ is a negative quantity, and as such, can have no square root; therefore the two roots of the quadratic equation $x^2+rx+rr-3a^2=0$, and consequently two of the roots of the cubic equation proposed, will be impossible; therefore if b be greater than a, or less (that is, more negative) than $-a$; in a word, if b^2 be greater than a^2, the equation $x^3-3a^2x=\pm 2a^2b$ can but have one possible root, which root will be greater than $2a$ or less than $-2a$, according as the absolute term of the equation is + or $-2a^2b$.

CASE 2.

Let now the equation be $x^3+3aax=2aab$: then it is plain that in this equation x cannot be negative; for if it was, both x^3 and $3aax$ would be negative; so that x^3+3aax could not possibly be equal to any affirmative quantity whatever; therefore if the equation $x^3+3aax=2aab$ has any roots, they must all be affirmative: but one root it has, by art. 430, which we will call r, and then we shall have $x-r$ a divisor of the quantity $x^3+3aax-2aab$; and so the demonstration might proceed as in the last article; but more briefly thus:

Let x be greater than r, and you will have x^3 greater than r^3, and $3aax$ greater than $3aar$, and x^3+3aax greater than r^3+3aar, and consequently greater than $2aab$, since $r^3+3aar=2aab$: in like manner if x be less than r, x^3+3aax will be less than $2aab$; therefore there is but one value of x, to wit r, that can make $x^3+3aax=2aab$; therefore the equation $x^3+3aax=2aab$ can have but one possible root, which will be affirmative or negative according as the absolute term of the equation is + or $-2aab$. *Q. E. D.*

444. For the better distinguishing the several cases of cubic equations, we considered them all under the form of $x^3\pm 3aax=\pm 2aab$. Let us now restore them to their primitive form $x^3\pm px=\pm q$, and then *The sum of the two last articles will amount to thus much; that the equation* $x^3-px=\pm q$, *if* $\frac{9qq}{4pp}$, *which is* bb, *be less than* $+\frac{p}{3}$, *which is* aa, *all the three roots will be possible; that the greatest of these roots will be affirmative or negative according as* q *is so, and that the signs of the other two will always be contrary to that of the greatest; that as to their situation, the greatest root will always lie between* $+2\sqrt{\frac{p}{3}}$ *and* $+\sqrt{p}$, *or* $-2\sqrt{\frac{p}{3}}$ *and* $-\sqrt{p}$; *that of the other two, that which is nearest to nothing will be between* 0 *and* $+\sqrt{\frac{p}{3}}$, *or* 0 *and* $-\sqrt{\frac{p}{3}}$, *and that the more remote will be between* $+\sqrt{\frac{p}{3}}$ *and* $+\sqrt{p}$,

or

or $-\sqrt{\frac{p}{3}}$ *and* $-\sqrt{p}$; *that in all other cases there will be but one possible root, which will be affirmative or negative according as the absolute term* q *is affirmative or negative; and particularly, that in the equation* $x^3-px=\pm q$, *the only possible root will be greater than* $2\sqrt{\frac{p}{3}}$, *or less than* $-2\sqrt{\frac{p}{3}}$.

This I say, is the amount of the two last articles: but here we may observe, that if in the equation $x^3-px=\pm q$, $\frac{+p}{3}$ be greater than $\frac{9qq}{4pp}$, we shall have by multiplication and division $\frac{+p^3}{27}$ greater than $\frac{qq}{4}$; whence we have another mark for determining the possibility or impossibility of the roots, which is this: *If the cube of* $\frac{p}{3}$ *affirmatively considered be greater than the square of* $\frac{q}{2}$, *all the three roots will be possible, otherwise not.*

Of the resolution of cubic equations.

445. *To resolve any cubic equation that hath but one possible root.*

Here are two cases; first, when the equation to be resolved is in this form, $x^3+px=+$ or $-q$; secondly, when it is in this form, $x^3-px=+$ or $-q$; in which last case, the cube of $\frac{p}{3}$ affirmatively taken is less than the square of $\frac{q}{2}$, because by the supposition the equation hath but one possible root.

Case I.

Let the equation be $x^3+px=\pm q$: then supposing m and n to be two unknown quantities, whereof m is the greater, substitute $m-n$ instead of x in the equation $x^3+px=+q$, because the root of that equation is affirmative, and you will have $m^3-3mmn+3mnn-n^3+pm-pn=q$. Now since as yet we have but one equation for determining the values of m and n, to wit $m-n=x$, we are at liberty to feign any other equation we please that may serve to render the equation more simple. Let us then suppose $3mn=p$, and we shall have $pm-pn=3mmn-3mnn$; and the equation will now stand thus, $m^3-3mmn+3mnn-n^3+3mmn-3mnn=q$, that is, $m^3-n^3=q$: but $3mn=p$ by the supposition; therefore $n=\frac{p}{3m}$, and $n^3=\frac{p^3}{27m^3}$: substitute now $\frac{-p^3}{27m^3}$ instead of $-n^3$ in the e-

quation

quation $m^3-n^3=q$, and you will have $m^3-\frac{p^3}{27m^3}=q$: multiply all by m^3, and you will have $m^6-\frac{p^3}{27}=qm^3$; whence $m^6-qm^3=\frac{p^3}{27}$, which is a sort of quadratic equation: compleat the square, and you will have $m^6-qm^3+\frac{qq}{4}=\frac{qq}{4}+\frac{p^3}{27}$: make $\frac{qq}{4}+\frac{p^3}{27}=ss$; and by extracting the square root, $m^3-\frac{q}{2}=\pm s$; whence $m^3=\frac{q}{2}\pm s$: of these two values of m^3, the latter, to wit $\frac{q}{2}-s$, in the present case is negative; for since $\frac{qq}{4}+\frac{p^3}{27}=ss$, we shall have $\frac{qq}{4}$ less than ss, and $\frac{q}{2}$ less than s, and $\frac{q}{2}-s$ less than nothing: make then $\frac{q}{2}+s=m^3$, and you will have $\frac{q}{2}-s=-n^3$, since $m^3-n^3=q$; and m will be the cube root of the affirmative quantity $\frac{q}{2}+s$, and $-n$ the cube root of the negative quantity $\frac{q}{2}-s$, and $m-n$ will be the root of the equation proposed. *If the equation be* $x^3+px=-q$, *first find the root of the equation* $x^3+px=+q$ *as above, and then change it's sign by art.* 439: *and the same must be observed in all other cases.* Thus in the present case the root of the equation $x^3+px=-q$ will be $n-m$.

CASE 2.

If the equation be $x^3-px=+q$, that is, if p be negative, to add the cube of $\frac{-p}{3}$ in this case will be the same as to subtract the cube of $\frac{+p}{3}$; and as ss is now equal to $\frac{qq}{4}-\frac{p^3}{27}$, s will be less than $\frac{q}{2}$, and $\frac{q}{2}-s$ will be an affirmative quantity; therefore $-n^3$ in the former case will now be changed into $+n^3$, and the root $m-n$ into $m+n$, and the following canon will include both cases, to wit, *To or from the square of* $\frac{q}{2}$ *add or subtract the cube of* $\frac{p}{3}$ *according as* p *is affirmative or negative in the equation proposed, and call the sum or remainder* ss: *make the cube root of* $\frac{q}{2}+s=m$, *and the cube root of* $\frac{q}{2}-s=\pm n$ *according as that quantity* $\frac{q}{2}-s$ *is affir-*

mative

mative or negative, and the root of the equation $x^3 \mp px = +q$ *will be* $m \pm n$ *according as* n *comes out affirmative or negative.*

For the ſake of what follows in the next article, it may not be amiſs to obſerve here that *In the equation* $x^3 \pm px = +q$, *the ſign of* n *in the root will always be contrary to that of* p: thus in the firſt caſe, where the equation was $x^3 + px = q$, the root was $m - n$, and in the ſecond, where the equation was $x^3 - px = q$, the root was $m + n$.

This ſecond caſe might alſo be demonſtrated from it's firſt principles like the firſt, thus: put $m+n$ for x in the equation $x^3 - px = +q$, and you will have $m^3 + 3m^2n + 3mn^2 + n^3 - pm - pn = q$: make $3mn = +p$, and exterminate n as before, and you will have the equation $m^3 + \frac{p^3}{27m^3} = q$; whence $m^6 - qm^3 = -\frac{p^3}{27}$, and $m^6 - qm^3 + \frac{qq}{4} = \frac{qq}{4} - \frac{p^3}{27}$: make $\frac{qq}{4} - \frac{p^3}{27} = ss$, and you will have $m^3 = \frac{q}{2} \pm s$; therefore ſince in this caſe $m^3 + n^3 = q$, if m^3 be made equal to $\frac{q}{2} + s$, we ſhall have $n^3 = \frac{q}{2} - s$; whence may be deduced the rule as above.

But here it muſt be obſerved, that if the cube of $\frac{p}{3}$ be greater than the ſquare of $\frac{q}{2}$, the quantity s in the ſecond caſe will be impoſſible, and then the equation cannot be reſolved this way, at leaſt not by poſſible numbers: not becauſe the deſired root becomes then impoſſible, for there are more poſſible roots in this caſe than in any other; but becauſe then the cubes m^3 and n^3 become impoſſible, and ſo their cube roots m and n cannot ſo eaſily be extracted: where this can be done, the impoſſibility of thoſe roots will be no hinderance in the application of the foregoing rule: as for inſtance, let $m = a + \sqrt{-b}$, and $n = a - \sqrt{-b}$, then it is plain that m and n are both impoſſible; and yet their ſum $m + n$, that is, the root of the equation will be a real quantity, as $2a$, the impoſſible parts deſtroying one another: but of this more hereafter. The very ſuppoſition upon which this *analyſis* is founded plainly ſhews, that the foregoing rule drawn from it cannot be univerſal, becauſe the ſuppoſition itſelf is not univerſally true. There it is ſuppoſed that the root x may be divided into two ſuch parts, as that the product of their multiplication may be $\frac{p}{3}$; but what if the root x ſhould be ſo ſmall a quantity as not to be capable of being ſo divided? why then this method of reſolving cubic equations muſt neceſſarily

ceſſarily become impracticable by poſſible numbers. When a number is divided into two parts, the product of their multiplication will be greateſt when the parts are equal; (ſee art. 111, obſ. 2d;) therefore $\frac{x}{2} \times \frac{x}{2}$ or $\frac{xx}{4}$ is the greateſt product that can ariſe from any two parts of x multiplied together: ſuppoſe now this greateſt product $\frac{xx}{4}$ to be leſs than $\frac{p}{3}$, and you will have xx leſs than $\frac{4p}{3}$, and x leſs than $2\sqrt{\frac{p}{3}}$: now that x will be leſs than $2\sqrt{\frac{p}{3}}$ in the caſe where all the roots are poſſible, and that in this caſe alſo the cube of $\frac{p}{3}$ will be greater than the ſquare of $\frac{q}{2}$, was ſufficiently demonſtrated in art. 442 and 444.

N. B. Though the rule here given for reſolving theſe two caſes of cubic equations be by *Cardan* himſelf aſcribed to one *Scipio Ferreus*, a noted Mathematician before his time, yet it is generally known by the name of *Cardan*'s rule.

EXAMPLE 1.

Let the equation to be reſolved be $x^3+30x=117$. Here $p=30$, $q=117$, the ſquare of $\frac{q}{2}=\frac{13689}{4}$, the cube of $\frac{p}{3}=1000=\frac{4000}{4}$, the ſum $ss=\frac{17689}{4}$, $s=\frac{133}{2}$, $\frac{q}{2}+s=125$, $\frac{q}{2}-s=-8$, $m=5$, $-n=-2$, $m-n$ or $x=3$, which will anſwer the condition of the equation.

EXAMPLE 2.

Let us try the more general rule in this ſimple caſe, to wit $x^3=q$. Here then $p=0$, $q=q$, $\frac{qq}{4}=\frac{qq}{4}$, $\frac{p^3}{27}=0$, the ſum $ss=\frac{qq}{4}$, $s=\frac{q}{2}$, $\frac{q}{2}+s=q$, $\frac{q}{2}-s=0$, $m=\sqrt[3]{q}$, $n=0$, $m-n$ or $x=\sqrt[3]{q}$.

EXAMPLE 3.

Let the equation be $x^3-36x=91$. Here $p=36$, $q=91$, $\frac{8281}{4}$, $\frac{p^3}{27}=1728=\frac{6912}{4}$, the difference $ss=\frac{1369}{4}$, $s=\frac{37}{2}$, $\frac{q}{2}+s$ 64, $\frac{q}{2}-s=27$, $m=4$, $n=3$, $m+n$ or $x=7$.

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EXAMPLE 4.

Let the equation be $x^3 - 12x = 16$. Here $p = 12$, $q = 16$, $\frac{qq}{4} = 64$, $\frac{p^3}{27} = 64$; the difference $ss = 0$, $\frac{q}{2} + s = 8$, $\frac{q}{2} - s = 8$, $m = 2$, $n = 2$, $m + n$ or $x = 4$, which is one root of the equation: but becaufe in this cafe the cube of $\frac{p}{3}$ is equal to the fquare of $\frac{q}{2}$, this equation will alfo have other two roots which will be negative and equal to one another, by art. 442; therefore each of the other two roots muft be -2, that both together may be able to balance the affirmative root 4 in taking away the fecond term.

EXAMPLE 5.

Let the equation be $x^3 + 21x = 50$. Here $p = 21$, $\frac{qq}{4} = 625$, $\frac{p^3}{27} = 343$, the fum $ss = 968$, $s = \sqrt{968}$, $\frac{q}{2} + s = 25 + \sqrt{968}$, $\frac{q}{2} - s = 25 - \sqrt{968}$: now the cube root of $25 + \sqrt{968}$ is $1 + \sqrt{8}$, and the cube root of $25 - \sqrt{968}$ is $1 - \sqrt{8}$ by art. 426, ex. 2; therefore $m = 1 + \sqrt{8}$, $-n = 1 - \sqrt{8}$, and $m - n$ or $x = 2$.

EXAMPLE 6.

Let the equation be $x^3 - 3x = 18$. Here $p = 3$, $q = 18$, $\frac{qq}{4} = 81$, $\frac{p^3}{27} = 1$, the difference $ss = 80$, $s = \sqrt{80}$, $\frac{q}{2} + s = 9 + \sqrt{80}$, $\frac{q}{2} - s = 9 - \sqrt{80}$: now the cube root $9 + \sqrt{80}$ is $\frac{3 + \sqrt{5}}{2}$, and the cube root of $9 - \sqrt{80}$ is $\frac{3 - \sqrt{5}}{2}$ by art. 426, ex. 3; therefore $m = \frac{3 + \sqrt{5}}{2}$, $n = \frac{3 - \sqrt{5}}{2}$ and $m + n$ or $x = 3$.

EXAMPLE 7.

Let the equation be $x^3 + x = 10$. Here $p = 1$, $q = 10$, $\frac{qq}{4} = 25$, $\frac{p^3}{27} = \frac{1}{27}$; the fum $ss = \frac{676}{27} = \frac{676 \times 3}{81}$; $s = \frac{26}{9} \times \sqrt{3} = \frac{13}{9} \times \sqrt{12}$, $\frac{q}{2} + s = 5 + \frac{13}{9}\sqrt{12}$, $\frac{q}{2} - s = 5 - \frac{13}{9}\sqrt{12}$: now m the cube root of $5 + \frac{13}{9}\sqrt{12}$

$\frac{13}{9}\sqrt{12}$ is $1+\frac{\sqrt{12}}{3}$, and $-n$ the cube root of $5-\frac{13}{9}\sqrt{12}$ is $1-\frac{\sqrt{12}}{3}$ by art. 426, ex. 4; therefore $m-n$ or $x=2$.

EXAMPLE 8.

Let the equation be $x^3+6x=36\sqrt{3}$. Here $p=6$, $q=36\sqrt{3}$, $\frac{qq}{4}=972$, $\frac{p^3}{27}=8$, the sum $ss=980$, $s=\sqrt{980}$, $\frac{q}{2}+s=18\sqrt{3}+\sqrt{980}=\sqrt{972}+\sqrt{980}$, $\frac{q}{2}-s=\sqrt{972}-\sqrt{980}$: now m or the cube root of $\sqrt{972}+\sqrt{980}$ is $\sqrt{3}+\sqrt{5}$, and $-n$ the cube root of $\sqrt{972}-\sqrt{980}$ is $\sqrt{3}-\sqrt{5}$ by art. 426, ex. 1; therefore $m-n$ or $x=2\sqrt{3}$.

This last equation, to wit $x^3+6x=36\sqrt{3}$, might also have been thus resolved: make $y\times\sqrt{3}=x$, and you will have $x^3=3\sqrt{3}\times y^3$, $6x=6\sqrt{3}\times y$, and the equation will be $3\sqrt{3}\times y^3+6\sqrt{3}\times y=36\sqrt{3}$: divide the whole equation by $3\sqrt{3}$, and you will have $y^3+2y=12$, where $p=2$, $q=12$, $\frac{qq}{4}=36$, $\frac{p^3}{27}=\frac{8}{27}$, the sum $ss=\frac{980}{27}$, $s=\sqrt{\frac{980}{27}}$, $\frac{q}{2}+s=6+\sqrt{\frac{980}{27}}$, $\frac{q}{2}-s=6-\sqrt{\frac{980}{27}}$, $m=1+\frac{\sqrt{15}}{3}$, $-n=1-\frac{\sqrt{15}}{3}$, $m-n$ or $y=2$, $y\times\sqrt{3}$ or $x=2\sqrt{3}$.

446. Whenever a cubic equation of the foregoing form has a whole number for it's root, the rule in the last article will certainly find it, either with or without the extraction of the cube root of a binomial; for no binomial will offer itself in this case that doth not admit of a binomial cube root, and that root may be found by the directions given in art. 426: yet after all I must own, that I cannot but think this way of resolving a cubic equation, by extracting the cube root of a binomial, to be somewhat unnatural, as engaging us in enquiries that have not a direct tendency to the point we would be at; therefore I should rather chuse to recommend the following *praxis*, which to me appears much more simple and direct.

Retaining the notation of the last article, and supposing what was there demonstrated, m *is the cube root of the quantity* $\frac{q}{2}+s$: *take then the nearest whole number to that root, or the nearest mixt number consisting of a whole number and some simple fraction annexed, and call it* m; *then since* $3mn=p$, *you will have* $n=\frac{p}{3m}$, *and the root of the equation will be* $m\pm\frac{p}{3m}$ *nearly;*

ly; that is, if the equation be $x^3+px=q$, *the root will be* $m-\frac{p}{3m}$; *but if the equation be* $x^3-px=q$, *the root will be* $m+\frac{p}{3m}$, *the sign of* p *in the root being always contrary to what it is in the equation by an observation in the last article. Take now the nearest whole number to the quantity* $m\pm\frac{p}{3m}$, *and that will be the root of the equation if it has a rational root; but if it has not, that number will however be the nearest whole number to the root, and will be a proper* basis *to found an approximation upon, whereby we may approach as near the true root as we please, as will be shewn hereafter.*

N. B. The extraction of the cube root *m* will be made somewhat easier by an observation at the end of the third example in art. 426, which see.

Examples of this method of resolving cubic equations.

EXAMPLE 1.

Let the equation be $x^3-20x=96$. Here $p=20$, $q=96$, $\frac{qq}{4}=2304$, $\frac{p^3}{27}=\frac{8000}{27}=296$ nearly, for the utmost exactness is not here required; therefore the difference $ss=2008$, $s=45$, $\frac{q}{2}+s=93$, $m=4\frac{1}{2}$, $\frac{p}{3m}=\frac{3}{2}-\frac{1}{54}$, $m+\frac{p}{3m}=6-\frac{1}{54}$; therefore the nearest whole number to $m+\frac{p}{3m}$ is 6: with this number 6 I try the equation, and it succeeds; for if x be put equal to 6, you will have $x^3-20x=96$.

EXAMPLE 2.

Let the equation be $x^3+x=10$, which is the same with that in the seventh example of the last article. Here then $p=1$, $q=10$, $\frac{qq}{4}=25$, $\frac{p^3}{27}=\frac{1}{27}$, the sum $ss=25+\frac{1}{27}$; but because the fraction $\frac{1}{27}$ is but small in respect of the number 25 joined with it, I make $ss=25$, whence $s=5$, $\frac{q}{2}+s=10$, $m=2$, $\frac{p}{3m}=\frac{1}{6}$ and $m-\frac{p}{3m}=2-\frac{1}{6}$, to which the nearest whole number is 2; therefore I try the equation with the number 2, and it succeeds.

EXAMPLE 3.

Let the equation be $x^3 - 16x = 40$. Here $p = 16$, $q = 40$, $\frac{qq}{4} = 400$, $\frac{p^3}{27} = \frac{4096}{27} = 152$, the difference $ss = 248$, $s = 16$, $\frac{q}{2} + s = 36$, $m = 3.3$ or $3\frac{1}{3}$ nearly, $\frac{p}{3m} = 1.6$; therefore $m + \frac{p}{3m} = 4.9$, to which the nearest whole number is 5; therefore I try the equation with the number 5, but it does not succeed; whence I conclude that this equation has no rational root, but that 5 is the nearest whole number to it.

If the root of a cubic equation be required true to seven places without any approximation, the best way will be to extract the cube root by the help of logarithms, as in the following example.

EXAMPLE 4.

Let the equation be $y^3 - 2y = 5$. Here the square of $\frac{q}{2}$ is $\frac{25}{4} = 6.25$, and the cube of $\frac{p}{3}$ is $\frac{8}{27} = .296296296296$; whence we have the difference $ss = 5.953703703704$, and $s = 2.440021$; and $\frac{q}{2} + s$ or $m^3 = 4.940021$; log. m^3 (that is, the logarithm of m^3) $= 0.6937288$; whence log. $m = 0.2312429$; log. $3m = 0.7083642$; log. $\frac{p}{3m} = -1.5926658$; $m = 1.703111$; $\frac{p}{3m} = .3914405$; $m + \frac{p}{3m}$ or $y = 2.0945515$. Nine of the first figures of the root are 2.09455148. See art. 458.

447. In the last article but one I observed, that the rule there given for resolving a cubic equation of this kind, $x^3 - px = \pm q$, subsists even in the case where the cube of $\frac{p}{3}$ exceeds the square of $\frac{q}{2}$; and that it might be applied here with the same success as in other cases, had we any certain rule for extracting the cube root of an impossible binomial; I mean a binomial, one part whereof is an impossible quantity. Give me leave to produce one instance of this out of an infinite number of others, if it be only to shew the irresistible force and immutable nature of truth, which is able to penetrate even through impossibilities, and can never be so distressed or so severely tried, as to be found inconsistent with herself. Let then the equation to be resolved be $x^3 - 15x = 4$: that this equation does not belong to any of the cases hitherto considered is plain, because p has here

here a negative sign before it, and 125 the cube of $\frac{p}{3}$ is greater than 4 the square of $\frac{q}{2}$: here then $ss=4-125=-121=+121\times-1$; therefore $s=11\times\sqrt{-1}$; therefore $\frac{q}{2}+s=2+11\sqrt{-1}$, and $\frac{q}{2}-s=2-11\sqrt{-1}$. Now the cube root of the impossible binomial $2+11\sqrt{-1}$ is $2+\sqrt{-1}$, which I thus demonstrate: the cube of $\overline{a+b}$ is $a^3+3aab+3abb+b^3$; make $a=2$, and $b=\sqrt{-1}$, and you will have $a^3=8$, $3aab=12\sqrt{-1}$, $3abb=6\times-1=-6$, and $b^3=-\sqrt{-1}$, and the sum of these terms is $2+11\sqrt{-1}$; therefore the cube of $2+\sqrt{-1}$ is $2+11\sqrt{-1}$; therefore *e converso* the cube root of $2+11\sqrt{-1}$ is $2+\sqrt{-1}$, and the cube root of $2-11\sqrt{-1}$ is $2-\sqrt{-1}$; so that in this case $m=2+\sqrt{-1}$, $n=2-\sqrt{-1}$, and $m+n$ or $x=4$, which is true; for if x be made equal to 4, you will have $x^3-15x=4$: but this is a case wherein all the three roots of the equation are possible; therefore if the other two possible roots are desired, all the equation must be thrown to one side, and then that side must be divided by $x-4$ thus. If $x^3-15x=4$, we shall have $x^3-15x-4=0$; divide $x^3-15x-4=0$ by $x-4=0$, and the quotient will be $xx+4x+1=0$; which equation when resolved, gives $x=-2+\sqrt{3}$ or $-2-\sqrt{3}$: so that the three roots of the equation proposed are 4, $-2+\sqrt{3}$, and $-2-\sqrt{3}$, any of which being substituted for x in the equation, will give $x^3-15x=4$.

It may perhaps be demanded, since there is no known rule for extracting the cube root of an impossible binomial, how I came to discover that the cube root of $2+11\sqrt{-1}$ was $2+\sqrt{-1}$: but my answer to this is, that no such discovery was made; that this cube root was known by composition and not by resolution thus: I assume two impossible binomials whose impossible parts are equal and contrary to one another, as $2+\sqrt{-1}$, and $2-\sqrt{-1}$; and then making $2+\sqrt{-1}=m$, and $2-\sqrt{-1}=n$, I multiply these together, and find $mn=5$: for $\overline{a+b}\times\overline{a-b}$ gives $aa-bb$; make $a=2$ and $b=\sqrt{-1}$ and you will have $aa=4$, $b^2=-1$, and $-b^2=+1$; whence $aa-bb=5$: since then $mn=5$, we shall have $3mn=15$. Again, $m^3=2+11\sqrt{-1}$ as above, and $n^3=2-11\sqrt{-1}$; therefore $m^3+n^3=4$. Now from the *analysis* in the last article but one it appears, that $3mn$ is equal to the coefficient p in the equation to be resolved, and that in the case where p hath a negative sign before it, m^3+n^3 is equal to the absolute term q; therefore I form an equation making $p=15$ and $q=4$, and it will stand thus, $x^3-15x=4$; and this equation may now be easily resolved again, since the cube roots of the impossible binomials $2+11\sqrt{-1}$ and $2-11\sqrt{-1}$ are known.

 Whether

Whether any one has found a way of extracting the cube root of an impossible binomial, I know not; most authors make no mention of it, and those that do, pass it slightly over: Dr. *Wallis* indeed has made some faint attempts towards it in a tentative way, but with little success.

448. I come now to consider more particularly this last case of cubic equations, to wit $x^3 - px = \pm q$, where the cube of $\frac{p}{3}$ affirmatively considered is greater than the square of $\frac{q}{2}$; for which purpose it will be sufficient only to consider the equation $x^3 - px = +q$, according to art. 439. Now though this equation cannot be resolved accurately in a direct way, even when the root is rational, yet indirectly it may; and when the root is irrational, the following rule will find it true in three of it's first figures at least, if not in more; so that whenever the root is required to a greater degree of exactness, this will be abundantly more than sufficient to found an approximation upon according to *Newton*'s rule hereafter to be explained.

Make $\sqrt{\frac{p}{3}} = a$, $\frac{aq}{2p} = b$, $\frac{11p}{144} = c$, *and* $\sqrt{b+c} = d$: *I say then that the affirmative root of the equation proposed will be* $\frac{5}{4}a + d$ *very near.*

In art. 442 *and* 444 *it was demonstrated, that this affirmative root never ascends higher than* $2\sqrt{\frac{p}{3}}$, *that is, according to our present notation, never higher than* 2a, *nor descends lower than* $\sqrt{p}$ *or* $a\sqrt{3}$; *that in the former case, the square of* $\frac{q}{2}$ *will be equal to the cube of* $\frac{p}{3}$; *whence* q *will be equal to twice the square root of* $\frac{p^3}{27}$ *or the square root of* $\frac{4p^3}{27}$; *and that in the latter case* q *will be equal to* 0; *so that* q *has it's limits as well as the root of the equation. Moreover it will appear from the demonstration of this rule, that when the affirmative root rises to it's highest limit* 2a, *the rule here given will be exact, and that it will fail most when the affirmative root falls into it's lowest limit* $a\sqrt{3}$, *where* q = 0. Let us therefore try the rule in this most disadvantageous case, and let us see whether even in this case it will not give us a root true to three places.

Let the equation to be resolved be $x^3 - 3x = 0$. Here dividing by x, we have $xx - 3 = 0$; whence x the affirmative root of the equation is $\sqrt{3}$, that is, 1.732 &c, by which the foregoing rule may be tried: but before I can apply it I must take notice, that as in this case $a = 1$, we shall have $\frac{5}{4}a$ (which is the greatest part of the root sought) = 1.25; whence it appears, that the three first figures of this root will consist of one inte-

gral,

gral, and two decimal places; so that the terms defined in this rule, where they are not exact, need not to be computed to above two decimal places thus: $p=3$, $q=0$, $a=1$, $b=0$, $c=.23$, $d=.48$, $\frac{5a}{4}=1.25$; therefore $\frac{5a}{4}+d=1.73$, which is the root of the equation proposed, true to three places.

Again, let the equation be $x^3-3x=.929203$. Now when $p=3$ as in the present case, the highest limit q can arrive at is $+2$, and the lowest 0; therefore in the equation last proposed, q is almost in the middle between it's two limits, but somewhat nearer the lowest: let us try however whether in this case our rule will not give us the root of this equation true to four places, observing here as before, that of the four first places of this root, one is integral and the other three decimal, and consequently that the quantities q, a, b, c, d, where they are not exact, must be computed to three decimal places thus: $p=3$, $q=.929$, $a=1$, $b=.155$, $c=.229$, $d=.620$, $\frac{5}{4}a=1.250$, $\frac{5}{4}a+d=1.870$: the true root is 1.87; therefore our rule finds the root of this equation true to four places.

Again, let the equation of the last article be proposed, to wit, $x^3-15x=4$. Here as $p=15$, the highest limit q can arrive at is $\sqrt{500}$, that is, $22+$, and the lowest is nothing; therefore in this equation, q is so near its lowest limit that we must not expect to have the root true to above three places, whereof one will be integral, and the other two decimal, because in this case $a=\sqrt{5}$. Here then $p=15$, $q=4$, $a=2.24$, $b=.30$, $c=1.15$, $d=1.20$, $\frac{5}{4}a=2.80$, $\frac{5}{4}a+d=4.00$; the true root is 4 by the last article.

Lastly, let the equation be $x^3-8x=3$, where the limits of q are 8.7 and 0. Here $p=8$, $q=3$, $aa=2.666667$, $a=1.633$, $b=.306$, $c=.611$, $d=.958$, $\frac{5}{4}a=2.041$, $\frac{5}{4}a+d=2.999$, which is too little by an unit in the fourth place; for the true root is 3.

I observed before, that though an equation of this kind cannot be resolved in a direct way even when the root is rational, yet indirectly it may; for who can see the root thus found approaching so very near the number 3, without trying the number 3 itself? and if he does, he will find it to be the true root of the equation.

Being now to demonstrate the foregoing rule, I shall only observe beforehand that the affirmative root of a cubic equation is comprehended within much narrower limits than either of the other two, which is the reason why it is the most easily found: for betwixt the highest limit $2a$, and the lowest $a\sqrt{3}$ is little more than $\frac{1}{4}a$ difference; so that if $2a-e$ be put equal to the affirmative root sought, the quantity e when least may

be

be equal to nothing, and when greatest can never much exceed $\frac{1}{4}a$. This being observed, put $2a-e$ for the affirmative root x in the equation $x^3-px=q$, or $x^3-3aax=q$, and you will have $x^3=8a^3-12aae+6ae^2-e^3$, $-3aax=-6a^3+3aae$, and $x^3-3aax=2a^3-9aae+6aee-e^3=q$: now from what has been already observed, the quantity e will be but small in respect of $2a$ the other part of the root joined with it; and if so, it's cube e^3 is of so small account in the foregoing equation, that if it be thrown out, the equation will not be much affected by it: let it then be thrown out, and the equation will be $6aee-9aae+2a^3=q$, or $6aee-9aae=q-2a^3$: divide all by $6a$, and you will have $ee-\frac{3}{2}ae=\frac{q-2a^3}{6a}$: but $\frac{q}{6a}=\frac{aq}{6a^2}=\frac{aq}{2p}=b$, and $\frac{2a^3}{6a}=\frac{aa}{3}$; therefore $ee-\frac{3}{2}ae+\frac{9}{16}aa=b-\frac{1}{3}aa+\frac{9}{16}aa$: but $-\frac{1}{3}aa+\frac{9}{16}aa=\frac{11}{48}aa=\frac{11}{144}\times 3aa=\frac{11p}{144}=c$; therefore $ee-\frac{3}{2}ae+\frac{9}{16}aa=b+c$: extract the square root, and you will have $e-\frac{3}{4}a=\pm\sqrt{b+c}=\pm d$; whence $e=\frac{3}{4}a\pm d$: but e in the sense it is here taken, cannot be $\frac{3}{4}a+d$, because it can never much exceed $\frac{1}{4}a$; therefore e must be $\frac{3}{4}a-d$, and $2a-e$ or the affirmative root sought must be $\frac{5}{4}a+d$ very near; and nearer, as e approaches nearer to nothing, that is, as the affirmative root approaches nearer to its highest limit $2a$, or as q approaches to $\sqrt{\frac{4p^3}{27}}$.

DEFINITIONS.

449. All cubic equations comprehended under the two first cases, that is, where $x^3+px=\pm q$, or where $x^3-px=\pm q$, and the cube of $\frac{p}{3}$ affirmatively taken is less than the square of $\frac{q}{2}$; all these equations I say, may be resolved two ways, either grossly by the foregoing rules, and then more correctly by an approximation, or else at once by a more accurate extraction of the square and cube roots without any further correction, as was shewn at the latter end of art. 446; and this cannot be more expeditiously performed than by a good table of logarithms. And as these two cases may be resolved by a canon of logarithms, so may the third and last case be resolved by a canon of sines, as will be taught hereafter: but first it will be proper to acquaint the reader (if he knows it not already) what the sine of an arc or an angle is, and what is meant by a canon of sines.

The

The sine then of any arc of a circle, is a line drawn from either extremity of that arc, perpendicular to a diameter passing through the other extremity of the same arc. Thus in the circle *ABCD (Fig. 62,)* whose diameter is *AEC*, the sine of the arc *AB* is the line *BE*, which falls from *B* one extremity of that arc perpendicularly upon a diameter passing through *A* the other extremity: by which definition it appears that the same line *BE* is also the sine of the arc *BC*; for it falls from *B* one extremity of that arc, perpendicularly upon a diameter that passes through *C* the other extremity of the same arc: so that whatever line is the sine of any arc, the same will also be the sine of its complement to a semicircle. It follows also from this definition, that *The sine of any arc is half the chord of twice that arc:* for if *BE* be produced through *E* till it again meets the circle in *D*, the line *BE* will be half the line *BD*; and that line *BD* will be the chord of the arc *BAD* which is twice the arc *BA*; and the same line *BD* will be also the chord of the arc *BCD* which is twice the arc *BC*.

The sine of an angle is nothing else but the sine of the arc that measures it: as if *F* be the center of the circle *ABCD*, and the line *BF* be drawn, the line *BE* may be said to be the sine of the angle *AFB*, because it is the sine of the arc *AB*; and it may also be said to be the sine of the angle *BFC*, because it is the sine of the arc *BC*.

A canon of sines is a table orderly exhibiting the sines of all arcs from one minute to ninety degrees: that is, supposing the *radius* of every circle to be divided into ten millions of equal parts, the canon expresses how many of these parts are contained in the sine of every degree and minute of a degree, from one minute to 90 degrees: and they need go no further, because (as I observed before) whatever line is the sine of any arc, the same will also be the sine of its complement to a semicircle; therefore if I want the sine of an arc of 150 degrees, I must look in the tables for the sine of 30 degrees: now over against 30 degrees in the canon of sines I find this number, 5000000, which shews that if the *radius* of a circle be divided into 10000000 of equal parts, the sine of 30 degrees will contain 5000000 of those parts, and consequently that the sine of an arc or an angle of 30 or 150 degrees is equal to half the *radius*; and so of the rest.

Note. Whenever the sine of an arc is mentioned without the *radius*, the tabular sine must always be understood, whose *radius* is known to be 10000000.

Note also, that a degree is the 360th part of the circumference of any circle, and a minute the 60th part of a degree.

A LEMMA.

450. *In every quadrilateral figure inscribed in a circle, the rectangle under the diagonals is equal to both the rectangles under the opposite sides taken together. (Fig. 63.)*

Let

Let $ABCD$ be a quadrilateral figure inscribed in a circle whose diagonals are AC and BD: I say then that the rectangle $AC\times BD$ is equal to the two rectangles $AB\times CD$ and $AD\times BC$ taken together. For from the angle A to the diagonal BD draw the line AE, so as to make the angle BAE equal to the angle CAD; then if the intermediate angle CAE be added to both, you will have the angle CAB equal to the angle EAD: moreover the angles ABE and ACD are equal, as standing both on the same arc AD; and for a like reason the angle ADE is equal to the angle ACB as standing both on the same arc AB. This equality of angles gives us two pair of similar triangles, to wit ABE, ACD, and ADE, ACB: in the former similar triangles, to wit ABE and ACD we have AB to BE as AC to CD; whence $AC\times BE=AB\times CD$: in the other similar triangles, to wit ADE and ACB we have AD to DE as AC to CB; whence $AC\times ED=AD\times BC$. Join both these equations together, and you will have $AC\times BE+AC\times ED=AB\times CD+AD\times BC$: but the two rectangles $AC\times BE$ $+AC\times ED$ are equal to the rectangle $AC\times BD$; therefore $AC\times BD=AB$ $\times CD+AD\times BC$, that is, the rectangle under the diagonals is equal to both the rectangles under the opposite sides taken together. *Q. E. D.*

A LEMMA. (*Fig.* 64.)

451. *Let* ABC *be an equilateral triangle inscribed in a circle; and from any one of it's angles* A, *let a line as* ADE *be drawn, cutting the opposite side* BC *in* D, *and the circle in* E; *and join the chords* BE, CE: *I say then that the chord* AE *will be equal to the sum of the two chords* BE *and* CE *put together.*

For by the foregoing lemma, the quadrilateral figure $ABEC$ gives the following equation, $AE\times BC=AB\times EC+AC\times EB$: but the three sides of the triangle ABC are equal by the supposition: cancel them then in the foregoing equation, and you will have $AE=BE+EC$. *Q. E. D.*

N. B. This lemma might have been demonstrated without the help of the former; but we shall have occasion for them both.

A PROBLEM. (*Fig.* 65, 66.)

452. *Let* ABCD *be an arc of a given circle whose diameter is* AF, *and let the arc* AB *be a third part of the whole arc* ABCD: *It is required, having given the chord of the arc* ABD, *to find the chord of the arc* AB; *and that, whether the arc* ABD *be less than a semicircle as in the sixtyfifth figure, or greater as in the sixtysixth; the notation, the reasoning and the conclusion in both cases being the same.*

Set off the arc BC equal to the arc BA, so that the three arcs AB, BC, CD may be equal to one another: also from F the end of the diameter,

set

set off on each side the arcs FE and FG, equal each to the arc AB; and then join the chords AB, BC, CD, DA, AC, BD; as also the chords AE, EF, FG, GA, GE: call the known diameter AF $2a$, and the known chord AD $2b$; call the chord AB or BC or CD or EF or FG x, and the chord AC or BD or EG y: and because the triangle AEF in a semicircle is right-angled at E, we shall have $AE^2 = AF^2 - FE^2 = 4aa - xx$, and AE or $AG = \sqrt{4aa - xx}$. These positions being observed, the two quadrilateral figures $ABCD$ and $AEFG$ furnish the two following equations by art 450, to wit, $AC \times BD = AB \times CD + AD \times BC$, that is, $yy = xx + 2bx$; and $AF \times EG = AE \times FG + AG \times EF$, that is, $2ay = \sqrt{4a^2 - x^2} \times 2x$; whence $y = \frac{\sqrt{4aa - xx} \times x}{a}$, and $yy = \frac{4aaxx - x^4}{aa}$: but in the former equation we had $yy = xx + 2bx$; therefore $\frac{4aaxx - x^4}{aa} = xx + 2bx$: multiply by aa and divide by x, and you will have $4aax - x^3 = aax + 2aab$; and by transposition $3aax - x^3 = 2aab$ or $x^3 - 3aax = -2aab$: whence it appears at last, that the chord AB will be one of the affirmative roots of this cubic equation $x^3 - 3aax = -2aab$; and if so, then the chord of a third part of the other arc on the other side AD must be the other affirmative root of the same equation: for whether x be put for the chord of a third part of one arc or the other, the equation will be the same, and therefore must equally respect both chords: this will appear by applying the same reasoning to both figures: but here we are to take notice, that in the equation $x^3 - 3aax = -2aab$, the quantity b affirmatively considered must not be greater than a; for then $2b$ or the chord AD would be greater than $2a$ or the diameter AF.

A PROBLEM.

453. *Allowing the trisection of a circular arc, it is required to construct a cubic equation of this form* $x^3 - px = \pm q$, *where the cube of* $\frac{p}{3}$ *is supposed to be greater (or not less) than the square of* $\frac{q}{2}$.

Throw the equation into this form $x^3 - 3aax = \pm 2aab$ by art. 440; then since $\frac{p^3}{27}$ is greater than $\frac{qq}{4}$, that is, since a^6 is greater than a^4bb, we shall have aa greater than bb, and a greater than b.

With the radius a *describe a circle* (Fig. 67,) *wherein let be inscribed the chord* AD = 2b; *take the arc* AB *equal to a third part of the whole arc* ABD, *and from the angle* B *inscribe in the circle the equilateral triangle*

 BHK,

BHK, *and join the chords* AB, AH, AK: *I say then that these three chords with proper signs before them will be the three roots of the equation proposed.* For first, let the equation be $x^3 - 3aax = -2aab$: then it is plain from the last article, that the chord AB, that is, a chord of a third part of the arc ABD will be one of the affirmative roots of the equation proposed, and that the chord of a third part of the arc AKD will be the other: but the chord AK is the chord of a third part of the arc AKD, which I thus demonstrate: The two arcs ABD and AKD make both together an entire circumference, and the two arcs AB and AK make up a third part of that circumference by the construction: but the arc AB is a third part of the arc ABD by the construction; therefore the arc AK is a third part of the arc AKD: therefore the two chords AB and AK will be the two affirmative roots of the equation proposed, and their sum with it's sign changed will be the third root, because the second term of the equation is wanting: but the sum of the two chords AB and AK is AH by art. 451; therefore the three roots of the equation proposed are the chords $+AB$, $+AK$ and $-AH$. Therefore secondly, if the equation be $x^3 - 3aax = +2aab$, it's three roots will be the chords $+AH$, $-AB$ and $-AK$.

N. B. In this construction, the quantities a, b and x are supposed either to be lines, or to be represented by lines.

A PROBLEM.

454. *Supposing all things as in the last article, let it now be required to resolve the foregoing equation by the help of a canon of sines.* (Fig. 67.)

Let p represent in degrees and minutes the arc whose sine is b, that is, half the arc ABD; then we shall have the arc $ABD = 2p$, the arc $AB = \frac{2p}{3}$, and the chord AB equal to twice the sine of the arc $\frac{p}{3}$: again, since the arc BAK is a third part of the whole circumference, that is, $\frac{360}{3}$, we shall have the arc $AK = \frac{360 - 2p}{3}$, and the chord AK equal to twice the sine of the arc $\frac{180 - p}{3}$: lastly, the arc ABH is equal to $\frac{360 + 2p}{3}$, and therefore the chord AH is twice the sine of the arc $\frac{180 + p}{3}$: but the arc $180 - p$ is the complement of the arc p to a semicircle, and the arc $180 + p$ is the complement of the arc $180 - p$ to an entire circle: find therefore, in a circle whose *radius* is a, three arcs, p, q and r, whereof p is the arc whose sine is b, q is the complement of p to a semicircle,

ſemicircle, and r the complement of q to an entire circle, and the three roots of the equation propoſed will be the double ſines of the arcs $\frac{p}{3}$, $\frac{q}{3}$ and $\frac{r}{3}$, their ſigns being determined as in the laſt article.

N. B. The more exactly the arc p is taken, the more exact will be the roots; but if there be any error in taking that arc, (and ſome there muſt be,) it will affect the ſine of $\frac{p}{3}$ moſt, and the ſine of $\frac{r}{3}$ leaſt; therefore as there is no occaſion to find above one of theſe roots trigonometrically, it will be moſt convenient to find the ſine of the arc $\frac{r}{3}$.

Before I proceed to give an example of this caſe, I muſt advertiſe the young learner, that in a table of ſines, we have not only the ſines of all arcs that can be expreſſed in degrees and minutes to 90 degrees, but their logarithms alſo, placed over againſt them, to ſave the trouble of taking them out of a table of logarithms; and the logarithm of the ſine of 90 degrees or of the *radius* is 10.0000000.

This premiſed, let the equation to be reſolved be $x^3-117x=324$: here $a=\sqrt{39}$, and $b=\frac{54}{13}$; therefore in this caſe the *radius* of the circle $ABHK$ is $\sqrt{39}$, and the ſine of the arc p in this circle is $\frac{54}{13}$: but the arc p cannot be known by this ſine, becauſe we have no tables calculated for this *radius*. Since then the arc p can only be known by its tabular ſine, that muſt be found by the rule of proportion thus: as $\sqrt{39}$ the *radius* of the circle $ABHK$ whoſe logarithm is 0.7955323, is to the tabular *radius* whoſe logarithm is 10.0000000, ſo is $\frac{54}{13}$ the ſine of the arc p in the circle $ABHK$, whoſe logarithm is 0.6184504, to the tabular ſine of the ſame arc, whoſe logarithm is 9.8229181. This logarithm was found by adding together the logarithms of the ſecond and third terms of the proportion, and ſubtracting from the ſum the logarithm of the firſt.

Having thus got the logarithmic ſine of the arc p, the arc p itſelf is found thus: amongſt the logarithmic ſines, and over againſt 9.8228302, I find 41 degrees 41 minutes, and over againſt 9.8229721, 41 degrees 42 minutes; therefore the arc p lies between 41 degrees 41 minutes, and 41 degrees 42 minutes; and to find it, I take not only the difference of the two neareſt logarithmic ſines above mentioned that lie on each ſide the ſine of p, but alſo the difference between the ſine of p and the neareſt leſs tabular ſine: the former difference I find to be 1419, and the latter 879; then I ſay, as 1419, the difference between the two neareſt ſines on each ſide the ſine of p, is to one minute, the difference of their arcs, ſo is 879, the difference between the ſine of p and the neareſt leſs tabular ſine, to

.619, the difference between the arc p and the neareſt leſs tabular arc: ſince then the neareſt leſs tabular arc is 41 degrees 41 minutes, the arc p will be 41 degrees 41.619 minutes. Subtract now the arc p from a ſemicircle, that is, from 180 degrees 00 minutes, and you will have the arc $q=138$ degrees 18.381 minutes; ſubtract again this arc q from an entire circle, that is, from 360 degrees 00 minutes, and you will have the arc $r=221$ degrees 41.619 minutes; whence $\frac{r}{3}=73$ degrees 53.873 minutes. Now before we can find the ſine of the arc $\frac{r}{3}$ in the circle $ABHK$, we muſt find it's tabular ſine thus: the logarithmic ſine of 73 degrees 53 minutes is 9.9825871, and the logarithmic ſine of 73 degrees 54 minutes is 9.9826236, and their difference is 365: therefore I ſay, as one minute, the difference between the two tabular arcs on each ſide the arc $\frac{r}{3}$, is to 365, the difference between the ſines of thoſe arcs, ſo is .873, the difference between the arc $\frac{r}{3}$ and the neareſt leſs tabular arc, to 319, the difference between the ſine of the arc $\frac{r}{3}$ and the neareſt leſs tabular ſine: therefore the tabular ſine of the arc $\frac{r}{3}$ is 9.9826190.

I now come to find the ſine of the ſame arc $\frac{r}{3}$ in the circle $ABHK$ thus: as the tabular *radius*, whoſe logarithm is 10.0000000, is to $\sqrt{39}$ the *radius* of the circle $ABHK$, whoſe logarithm is 0.7955323, ſo is the tabular ſine of the arc $\frac{r}{3}$, whoſe logarithm is 9.9826190, to the ſine of the ſame arc in the circle $ABHK$, whoſe logarithm is 0.7781513. This work muſt alſo be performed by logarithms, but the intermediate proportions muſt be wrought the common way.

Having now got 0.7781513 the logarithmic ſine of the arc $\frac{r}{3}$ in the circle $ABHK$, I take the natural number belonging to it out of the table of logarithms, and find it to be 6.000000; and the double of this natural ſine is the root of the equation, to wit 12.000000. I try with the number 12, and it ſucceeds.

This root 12 being found, the other two roots will be eaſily obtained by diviſion as uſual, thus: according to art. 431 and 432, throw the equation propoſed $x^3-117x=324$ to one ſide, and then divide $x^3-117x-324$

-324 by $x-12$, and the quotient will be $x^2+12x+27=0$; resolve this equation, and you will have -9 and -3 for the roots; therefore the three roots of the equation proposed are $+12$, -9 and -3.

A PROBLEM.

455. *To find two mean proportionals between any two given numbers.*

Let a and b be the two given extremes, and let x and y be the two mean proportionals sought, that is, let a, x, y and b be continual proportionals: then since a is to x as x is to y, and as y is to b; the fractions $\frac{a}{x}$, $\frac{x}{y}$ and $\frac{y}{b}$ will be all equal, and we shall have $\frac{a}{x} \times \frac{x}{y} \times \frac{y}{b}$, that is, $\frac{a}{b} = \frac{a^3}{x^3} = \frac{y^3}{b^3}$: the first equation $\frac{a}{b} = \frac{a^3}{x^3}$ gives $x=\sqrt[3]{a^2b}$, and the second equation $\frac{a}{b} = \frac{y^3}{b^3}$ gives $y=\sqrt[3]{ab^2}$. In words thus: *Multiply the square of the greater extreme into the less, and the square of the less into the greater, and the cube roots of the two products will be the two mean proportionals sought.* As for instance, let it be required to find two mean proportionals between the numbers 27 and 8: now the square of 27 is 729, which multiplied by 8 the other extreme, gives 5832, whose cube root is 18 the greater mean: again, the square of 8 is 64, which multiplied by 27 the other extreme, gives 1728, and the cube root of this is 12 the lesser mean: so that the two mean proportionals sought are 18 and 12, and the numbers 27, 18, 12 and 8 are in continual proportion, the common ratio being that of 3 to 2.

When the two middle proportionals sought are irrational numbers, they will best be found by their logarithms thus: to twice the logarithm of the greater extreme add the logarithm of the lesser extreme, and a third part of their sum will be the logarithm of the greater mean sought: again, to twice the logarithm of the lesser extreme add the logarithm of the greater, and a third part of their sum will be the logarithm of the lesser mean sought.

For the better understanding the following article, it may be proper to take notice, that this invention of two mean proportionals between two given extremes, is in other words called *the trisection of a ratio.* Thus, since the ratio of 27 to 18 is equal to that of 18 to 12, and this again is equal to the ratio of 12 to 8, the ratio of 27 to 8 is justly said to be divided into three equal ratios by the intervention of the terms 18 and 12; and so the ratio of 27 to 8 is said to be three times the ratio of 27 to 18, or three times the ratio of 3 to 2: and on the other hand, the ratio of 3 to 2, or of 27 to 18 is said to be a third part of the ratio of 27 to 8.

Mr.

Mr. Cotes's *method of resolving cubic equations considered and demonstrated.* (See Fig. 68.)

456. The late ingenious Mr. *Cotes* in his *Logometria*, page 29, considering all cubic equations as under this form $x^3 \pm 3aax = \pm 2aab$, resolves all the three cases by the help of a right-angled triangle *ABC*, right-angled at *A*, whereof two sides are always given, as representing the known quantities *a* and *b* in the equation, thus.

CASE 1.

If the equation be $x^3 + 3aax = \pm 2aab$, *make* $AB = a$, $AC = b$, *and let* m *and* n *be two mean proportionals between* $\overline{BC + CA}$ *and* $\overline{BC - CA}$: *then will* $m - n$ *be the only possible affirmative root, or* $n - m$ *the only possible negative root of the equation, according as the absolute term is* $+2aab$ *or* $-2aab$.

CASE 2.

If the equation be $x^3 - 3aax = \pm 2aab$, *and* a *be less than* b; *make* $AB = a$, $BC = b$, *and let* m *and* n *be two mean proportionals between* $\overline{BC + CA}$ *and* $\overline{BC - CA}$; *and* $m + n$ *will be the only possible affirmative root, or* $-m - n$ *the only possible negative root, according as the absolute term is* + *or* $-2aab$.

CASE 3.

If the equation be $x^3 - 3aax = \pm 2aab$, *and* a *be greater than* b; *make* $AB = b$, $BC = a$, *and let* m *be the sine of a third part of the sum of the two angles* A *and* B *and* n *the sine of a third part of their difference, in a circle whose radius is* BC; *and the three roots of the equation will be* $\overline{m + n}$, $-m$ *and* $-n$, *or* $+m$, $+n$ *and* $\overline{-m - n}$, *according as the absolute term is* + *or* $-2aab$.

The analogy of these three cases consists in this; that whereas the two first cases were resolved by the sum and difference of the two sides *BC* and *CA*, the last case is resolved by the sum and difference of their opposite angles *A* and *B*; and whereas in the two first cases, the roots were obtained by the trisection of a ratio, in the last they are had by the trisection of an angle; and indeed nothing is more common in nature than a transition from a section of a ratio to a like section of an angle; innumerable instances whereof we have in that incomparable treatise above cited.

If the curious reader would see a demonstration of these three cases, he must not scruple being sent back to the 445th article for the two first, and to the 453d and 454th for the last to refresh his memory, and the demonstration is as follows.

CASE

CASE 1.

Let the equation be $x^3+3aax=+2aab$; then will $p=3aa$, $q=2aab$, $\frac{qq}{4}=a^4bb$, $\frac{p^3}{27}=a^6$, $ss=a^6+a^4bb=a^4\times\overline{aa+bb}$: make $aa+bb=hh$, that is, in the triangle ABC right-angled at A, make $AB=a$, $AC=b$, and $BC=h$, and you will have $ss=a^4hh$, $s=aah$, $\frac{q}{2}+s$ or $m^3=aab+aah=aa\times\overline{h+b}$: but $aa=hh-bb=\overline{h+b}\times\overline{h-b}$; therefore $m^3=\overline{h+b}\times\overline{h+b}\times\overline{h-b}$. In like manner we ſhall find $\frac{q}{2}-s$ or $-n^3=\overline{b-h}\times\overline{h-b}\times\overline{h+b}$; and $+n^3=\overline{h-b}\times\overline{h-b}\times\overline{h+b}$; therefore m and n, whoſe difference is the root of the equation, are two middle proportionals between $h+b$ and $h-b$ by the laſt article, that is, between $\overline{BC+CA}$ and $\overline{BC-CA}$.

CASE 2.

Let the equation be $x^3-3aax=+2aab$, and let a be leſs than b: then we ſhall have $ss=a^4bb-a^6=a^4\times\overline{bb-aa}$: make $bb-aa=hh$, that is, in the abovementioned triangle, let $AB=a$, $BC=b$, and $CA=h$, and you will have $ss=a^4hh$, $s=aah$, $\frac{q}{2}+s$ or $m^3=aa\times\overline{b+h}$: but in this caſe $aa=bb-hh=\overline{b+h}\times\overline{b-h}$; therefore $m^3=\overline{b+h}\times\overline{b+h}\times\overline{b-h}$; and for a like reaſon, $n^3=\overline{b-h}\times\overline{b-h}\times\overline{b+h}$; therefore m and n, whoſe ſum is the root of the equation, are two middle proportionals between $\overline{b+h}$ and $\overline{b-h}$, that is, between $\overline{BC+CA}$ and $\overline{BC-CA}$.

CASE 3.

Laſtly, let the equation be $x^3-3aax=+2aab$, and let a be greater than b: then if in the abovementioned right-angled triangle ABC we ſuppoſe $AB=b$, and $BC=a$; the line BA or b will be the ſine of the angle C in a circle whoſe *radius* is a: for if upon the center C, and with the *radius* CB an arc of a circle be deſcribed, cutting the leg CA produced in D, BA will be the ſine of the arc BD, and conſequently of the angle C which it meaſures; therefore BD is the arc which in art. 454 we called p; therefore p expreſſes in degrees and minutes the quantity of the angle C: but the angle C is the difference of the two angles A and B, and may be expreſſed by $A-B$; therefore $p=A-B$: ſubtract this from 180 degrees, that

that is, from $2A$, and you will have $q=A+B$; but $\frac{p}{3}$ and $\frac{q}{3}$ are the two arcs or angles whose double sines in a circle whose *radius* is BC or a, or whose single sines in a circle whose radius is $2BC$, are the two negative roots of the equation; therefore if m be the sine of $\frac{A+B}{3}$, and n the sine of $\frac{A-B}{3}$ in a circle whose *radius* is $2BC$, the three roots of the equation will be $\overline{m+n}$, $-m$ and $-n$. The other cases where the absolute term $2aab$ is negative, are the reverse of these by art. 439.

Another method of resolving cubic equations, which finds all the three roots at once.

457. Having given this account of Mr. *Cotes*'s method of resolving cubic equations, I shall but just touch upon another, which nevertheless for its elegancy deserves a more distinct consideration than I can here allow it, having said so much already upon this subject. The method I here speak of is in the Philosophical Transactions, N°. 309, by the learned Mr. *John Colson*, a Gentleman whose great genius and known abilities in these sciences I shall always have in the highest admiration and esteem. Mr. *Colson*'s method is universal, but for brevity's sake I shall here apply it only to such cubic equations as have their second term wanting or taken away; nor will there be occasion to consider above one case of these, from whence a transition may be easily made to all the rest to which this method extends. *Let then the equation be* $x^3+px=q$; *and let* $\frac{q^2}{4}+\frac{p^3}{27}=ss$ *as in art.* 445; *and let the cube root of the irrational binomial* $\frac{q}{2}+s$ *be* $d+\sqrt{e}$: *I say then that the three roots of this equation will be, first* $2d$, *secondly* $-d+\sqrt{-3e}$, *and thirdly* $-d-\sqrt{-3e}$: for if $d+\sqrt{e}$ be the cube root of the binomial $\frac{q}{2}+s$, $d-\sqrt{e}$ will be the cube root of $\frac{q}{2}-s$; therefore $d+\sqrt{e}$ will be the same with m in art. 445, and $d-\sqrt{e}$ will be the same with $-n$; whence $m-n$, which is one of the roots of the equation proposed, will be $2d$. Now to find the other two roots, let us divide the quantity $x^3+px-q=0$ by $x-2d$ as usual, and the quotient will be $xx+2dx+4dd+p$; and the remainder, though it hath the form of something, will be actually nothing; see art. 443. Make then $xx+2dx+4dd+p=0$, and you will have $xx+2dx=-4dd-p$, and

$xx+2dx+dd=-3dd-p$: but if $d+\sqrt{e}=+m$, and $d-\sqrt{e}=-n$, we ſhall have $-mn=dd-e$, and $-3mn=3dd-3e$; but $3mn=p$ by art. 445; and therefore $-3mn$ or $+3dd-3e=-p$; ſubſtitute now $3dd-3e$ inſtead of $-p$, and you will have $xx+2dx+dd=-3e$; therefore $x+d=\pm\sqrt{-3e}$, and $x=-d+\sqrt{-3e}$, or $-d-\sqrt{-3e}$, which are the other two roots of the equation propoſed. If s and conſequently $\sqrt{e}$ be poſſible, $\sqrt{-3e}$ will be impoſſible, in which caſe theſe two laſt roots of the equation will be impoſſible; but if s and conſequently $\sqrt{e}$ be impoſſible, then $\sqrt{-3e}$ will be poſſible; whence it follows, that if the cube root of an impoſſible binomial could be extracted, this rule would extend itſelf to all the caſes of cubic equations. A few examples of this rule will be ſufficient.

EXAMPLE 1.

Let the equation be $x^3+3x=4$. Here $\frac{q}{2}=2$, $\frac{p}{3}=1$, $\frac{qq}{4}+\frac{p^3}{27}$ or $ss=5$, $s=\sqrt{5}$, $\frac{q}{2}+s=2+\sqrt{5}$; but the cube root of this irrational binomial $2+\sqrt{5}$ is $\frac{1+\sqrt{5}}{2}$; therefore $d=\frac{1}{2}$, $\sqrt{e}=\frac{1}{2}\sqrt{5}$, and $2d$, which is one root of the equation, will be 1: as for the other two roots, ſince $\sqrt{e}=\frac{\sqrt{5}}{2}$, we ſhall have $e=\frac{5}{4}$, $-3e=-\frac{15}{4}$, and $\sqrt{-3e}=\frac{1}{2}\sqrt{-15}$; therefore the other two roots are $-\frac{1}{2}+\frac{1}{2}\sqrt{-15}$, and $-\frac{1}{2}-\frac{1}{2}\sqrt{-15}$.

EXAMPLE 2.

Let the equation be $x^3-96x=576$. Here $\frac{q}{2}=288$, $\frac{p}{3}=32$, $\frac{qq}{4}-\frac{p^3}{27}$ or $ss=50176$, $s=224$, $\frac{q}{2}+s=512$, $\frac{q}{2}-s=64$, $d+\sqrt{e}=8$, $d-\sqrt{e}=4$; add the two laſt equations together, and you will have $2d=12$, ſubtract the latter from the former, and you will have $2\sqrt{e}=4$, whence $\sqrt{e}=2$, $e=4$, and $-3e=-12$; therefore the three roots of this equation are, firſt 12, ſecondly $-6+\sqrt{-12}$, and thirdly $-6-\sqrt{-12}$.

EXAMPLE 3.

Let the equation be $39x-x^3=70$, or $x^3-39x=-70$. Here $\frac{q}{2}=-35$, $\frac{p}{3}=13$, $\frac{qq}{4}-\frac{p^3}{27}$ or $ss=-972=18\times18\times-3$, $s=18\sqrt{-3}$,

$\frac{q}{2}+s=-35+18\sqrt{-3}$, whose cube root is $1-2\sqrt{-3}$, as will appear upon tryal; therefore $d+\sqrt{e}=1-2\sqrt{-3}$, $d=1$, $2d=2$, $+\sqrt{e}=-2\sqrt{-3}$, $e=+4\times-3=-12$, $-3e=+36$, $\sqrt{-3e}=6$, $-d+\sqrt{-3e}=5$, $-d-\sqrt{-3e}=-7$, and the three roots of the equation are $+2$, $+5$ and -7.

Newton's *method of approximation applied to the resolution of cubic equations.*

458. This method is best taught by example, and a better cannot be produced than that given by the author himself. Let it then be required to approach as near as we please to the root of this cubic equation $y^3-2y-5=0$: here, according to the author's direction, *We are first to find a number which differs not above a tenth part of itself from the true root*; (and how to find a much more accurate number than this, if occasion required, is abundantly shewn in the foregoing articles;) let that number be 2, and accordingly let it be written down as the first figure in the quotient that is to exhibit the root sought; let p represent the rest of the root, that is, let $2+p=y$ in the equation $y^3-2y-5=0$, and you will have $y^3-2y-5=p^3+6pp+10p-1=0$, as may be seen by the work; only there, this latter quantity $p^3+6pp+10p-1$ is inverted. Now the whole difficulty is reduced to this, to wit, to find the root of the equation $p^3+6pp+10p-1=0$: to effect this, we are to take notice that the quantity p must be a proper fraction, as not being above a tenth part of the number 2 by the supposition; therefore p^2 will be less than p, and p^3 less than p^2; therefore if in the last equation the terms wherein p^3 and p^2 are concerned are dropt, that is, the terms p^3 and $6p^2$, and if the two remaining terms $10p-1$ be made equal to nothing, the root of this equation will be nearly the same with the root of the whole: make then $10p-1=0$, and you will have $p=\frac{1}{10}$ or 0.1 which $.1$, must be written after 2 in the quotient: sometimes indeed it happens that three of the last terms are to be retained in order to obtain a proper value of p; but these cases shall be more distinctly considered afterwards. Let us now resume the former equation, to wit $p^3+6p^2+10p-1=0$; and having already found that p the root of this equation equals 0.1 nearly, let us make $p=0.1+q$, and we shall have $p^3+6p^2+10p-1=q^3+6.3q^2+11.23q+0.061=0$, as appears by the work: drop $q^3+6.3q^2$ as before, and you will have $11.23q+0.061=0$ very near; whence $q=-\frac{0.061}{11.23}$; reduce this fraction to a decimal, and you will have $q=-0.0054$ nearly, which being negative,

negative, must be written in the lower part of the quotient for the conveniency of subtracting it afterwards: and here it must be observed, that *In dividing the numerator of the fraction by the denominator, the division must be continued to as many places from the first significant figure inclusively, as there are places between this first figure and the first figure of the main quotient exclusively:* as for instance, the first significant figure arising from this division is 5 in the third decimal place, and the first figure in the main quotient is 2 in the place of units, and betwixt these two figures 2 and 5, and exclusive of both, are two places 0 and 0; therefore in dividing the numerator of the fraction by the denominator in order to reduce it to a decimal, the division must not be continued to above two places, to wit 5 and 4, and so q must be made equal to $-0.0054+r$ in the equation $q^3+6.3qq+11.23q+0.061=0$: and thus may you proceed to as many figures in the root as you please: there are indeed cases wherein the abovementioned division may be further continued than the rule here laid down allows; but it would not always be safe to assume such a liberty. If the next operation (where $-0.0054+r$ is made equal to q in the equation $q^3+6.3q^2+11.23q+0.061=0$) is intended to be the last, all those terms wherein r^3 and r^2 are concerned may be neglected; for they must be dropt at last without ever being resumed, and therefore they may as well be dropt at first; and then you will have, first $q^3=0.00008748r-0.000000157464$, secondly $6.3q^2=-0.06804r+0.000183708$, thirdly $11.23q=11.23r-0.060642$, fourthly $0.061=0.061$: add these equations together, and you will have $q^3+6.3q^2+11.23q+0.061=11.16204748r+0.000541550536=0$, whence $r=-\frac{0.000541550536}{11.16204748}$. As this operation was intended for the last, r^3 and the terms wherein r^2 was concerned were omitted, as above: but let us now see whether there do not still remain some parts which ought to be dropped, or rather, which ought not to have been taken notice of in the work: we have already two quotients, which both together will express the root sought to five places, to wit $+2.1000$ and -0.0054, therefore the first new figure gained by this operation will be in the sixth place; but betwixt the first and sixth places are included four intermediate ones; therefore according to the rule already laid down for continuing the division, four new places are as many as can be expected from this operation: if therefore in the fraction expressing the value of r, be thrown out of the numerator all after the four first significant figures, and out of the denominator all but four or five of the first, because the first figures of the denominator are but small, the fraction thus reduced will still exhibit the value of r as accurately as can be expected from this operation. Thus

Thus then we have $r = -\frac{0.0005416}{11.162}$, which when reduced to a decimal fraction is -0.00004852; place this to the other negative part of the quotient which is -0.0054, and the whole negative part will be -0.00544852; subtract this from the affirmative part which is $+2.10000000$, and the remainder 2.09455148 will be the root of the equation first proposed, true to nine places: see the work, wherein all beyond the third decimal in the several coefficients of r, and all beyond the seventh in the absolute terms are cancelled by lines drawn through them. But if this operation is not intended for the last, then all the cancelled terms must be taken into the account, and $-0.00004852+s$ must be made equal to r.

$y^3-2y-5=0.$		$+2.10000000$ -0.00544852 $+2.09455148$ &c $=y$
$2+p=y.$	$+y^3$ $-2y$ -5	$+8+12p+6p^2+p^3$ $-4-\ 2p$ -5
	sum	$-1+10p+6p^2+p^3$
$0.1+q=p.$	$+\ p^3$ $+\ 6p^2$ $+10p$ $-\ 1$	$+0.001+\ 0.03q+0.3q^2+q^3$ $+0.06\ +\ 1.2\ \ +6.0$ $+1.0\ \ +10.0$ -1.0
	sum	$0.061+11.23q+6.3q^2+q^3$
-0.0054 $+r=q.$	q^3 $+\ 6.3q^2$ $+11.23q$ $+\ 0.061$	$-0.0000001\cancel{57464}+\ 0.000\cancel{08748}r-\cancel{0.0162}r^2$ $+0.0001837\cancel{08}\ \ -\ 0.068\cancel{04}\ \ +\cancel{6.3}\ (+r^3$ $-0.060642\ \ +11.23$ $+0.061$
	sum	$+0.0005416\ \ +11.162r$
$-0.00004852+s=r.$		

459. The greatest difficulty in these approximations is in the first, I mean, in finding a single figure that shall be a sufficient approximation to the quantity p; for though not above one single figure is to be taken, yet if proper caution be not used, even that figure may be mistaken too. The rule therefore to be observed in this case is as follows: *Retaining the three last significant terms of the equation expressing the value of* p, *if the square of the middle of these three terms be equal to or greater than ten times the product*

product of the extremes, without any regard had to their signs, then it will be sufficient to make the two last of those three terms equal to nothing, and to take the first figure of the root of that simple equation for an approximation to the quantity p: *but if it happens otherwise, then make all the three terms equal to nothing, and extract the lesser of the two roots of that quadratic equation whether it be affirmative or negative, and the first figure of this root will be such a part of* p *as will be proper to found the next approximation upon.* Thus in art. 458, the equation expressing the value of p was $p^3+6p^2+10p-1=0$, whereof the three last significant terms were $6p^2+10p-1$; now the square of the middle term $10p$ was $100p^2$, and the product of the extremes without regard to their signs was $6p^2$: now as $100p^2$ was more than ten times $6p^2$, it was sufficient to make the two last terms $10p-1=0$, and so to make the first and only figure of the root of that simple equation the most considerable part of p; but if it had happened otherwise, then $6p^2+10p-1$ must have been made equal to nothing, and the first figure of the lesser root of that equation must have been taken for the nearest approach towards the quantity p.

The reason of this rule I shall easily deduce from the nature of a quadratic equation thus: let $ap^2=bp+c$ or $ap^2-bp-c=0$ represent any quadratic equation whatever, where the unknown quantity is p, and let us in the first place drop the first term ap^2, and then the equation will be reduced to this, $-bp-c=0$; where if the first part of the rule obtains, p will be equal to $-\frac{c}{b}$ nearly, as will presently be shewn: let us now take in all the terms of the equation, and make $ap^2-bp-c=0$, and we shall have $ap^2-bp=c$, and $p^2-\frac{bp}{a}=\frac{c}{a}$ and $p^2-\frac{bp}{a}+\frac{bb}{4aa}=\frac{bb}{4aa}+\frac{c}{a}=\frac{bb+4ac}{4aa}$, and $p-\frac{b}{2a}=\frac{-\sqrt{bb+4ac}}{2a}$; I say $-\sqrt{bb+4ac}$, because the lesser root is to be extracted; whence we have $p=\frac{b-\sqrt{bb+4ac}}{2a}$; but $\sqrt{bb+4ac}$, whether it be extracted according to *Newton*'s series, or the common way, is $b+\frac{2ac}{b}-\frac{2aacc}{b^3}$ &c; therefore $b-\sqrt{bb+4ac}=-\frac{2ac}{b}+\frac{2aacc}{b^3}$; therefore $\frac{b-\sqrt{bb+4ac}}{2a}=-\frac{c}{b}+\frac{acc}{b^3}$: thus then we have two different roots drawn from two different equations; one less exact, as $-\frac{c}{b}$ from a

simple

ſimple equation; the other more exact, as $-\frac{c}{b}+\frac{acc}{b^3}$ from a quadratic equation; and their difference is $\frac{acc}{b^3}$: now the queſtion is, what muſt be the relation of the three quantities a, b and c one to another, ſo that this difference may not affect the firſt figure of the common part $-\frac{c}{b}$; for wherever this happens, one equation will be as much for our purpoſe as the other: but this queſtion is ſoon anſwered; for it is very eaſy to ſee, that if the common part be equal to or greater than ten times the difference, then that difference cannot directly affect the firſt figure of the common part, but only thoſe that follow; ſo that the rule as it firſt comes out, ſtands thus: if $\frac{c}{b}$ be equal to, or greater than $\frac{10acc}{b^3}$, make the two laſt terms of the equation expreſſing the value of p equal to nothing, otherwiſe make the three laſt terms equal to nothing: now if $\frac{c}{b}$ be equal to, or greater than $\frac{10acc}{b^3}$, then multiplying both ſides into $\frac{b^3}{c}$, you will have bb equal to, or greater than $10ac$, and $bbpp$ equal to, or greater than $10acp^2$, agreeably to the rule firſt laid down. After the ſame manner it may be demonſtrated, that *If, of the three laſt terms of the equation expreſſing the value of* p, *the ſquare of the middle term be a hundred or a thouſand times greater than the product of the extremes, we may make the two laſt terms equal to nothing, and may take two or three of the firſt figures of the root of that equation, to found the next approximation upon.* The author tells us, that if not only the firſt figure of the root, but all the reſt be tried in the manner above deſcribed, that is, (if I apprehend him right,) if all the quantities p, q, r &c were found by the help of quadratic equations, the analyſt might then venture to take twice as many figures into the root at every operation, as he might do in a like caſe the other way: which is true; but then, if I am not miſtaken, that greater number of figures will be dearly bought by the laborious neſs of the work, unleſs the practitioner has ſagacity enough to foreſee and retrench all ſuch decimal places as cannot influence thoſe he intends to have in his concluſion; nay what is more, he that will make the beſt of this method, muſt have ſkill enough too, to avoid all ſuch ſuperfluous operations as are employed in finding numbers that afterwards deſtroy themſelves, and never enter into the main root. But I ſhall not here take upon me peremptorily to determine in matters where I have ſo little experience: I ſhall rather choose to ſubmit the

the whole to the practice of the learner, having thus introduced him into the method itself.

460. As in all these cases we have occasion to substitute quantities one for another, our author recommends another way of doing this, different from that already explained, and in some measure preferable to it, as being a readier way: I shall explain it by an example or two thus: let it be required to substitute $2+p$ for y in the equation $y^3-2y-5=0$: for our better direction in making this substitution, we must first enquire into the *genesis* or composition of the quantity y^3-2y-5 thus: first, to the root y should be added the coefficient of the second term, and then that sum should be multiplyed by y; but as there is no second term, multiply y itself by y, and the product will be y^2; add -2 the coefficient of the third term, and you will have $yy-2$; multiply this by y, and you will have y^3-2y; to this product add -5 the numeral part, and you will have the quantity proposed, to wit, y^3-2y-5. This being observed, we must now begin again, and treat the quantity $2+p$ just after the same manner as before we did it's equal y thus: $y=2+p$; now as the second term of the equation, and consequently its coefficient, is wanting, multiply the first side of this equation into y, and the latter side into its equal $2+p$, and you will have $yy=4+4p+p^2$; add -2, the coefficient of the third term, to both sides, and you will have $yy-2=2+4p+p^2$; multiply the first side by y, and the latter by $2+p$, and you will have $y^3-2y=4+10p+6p^2+p^3$; add lastly -5, the numeral part, to both sides, and you will have the quantity proposed, $y^3-2y-5=-1+10p+6p^2+p^3$, as in art. 458. Again, let it be required to substitute $0.1+q$ instead of p in the equation $p^3+6p^2+10p-1=0$: here $p=0.1+q$; add 6, the coefficient of the second term, to both sides, and you will have $p+6=6.1+q$; multiply the former side by p, and the latter by $0.1+q$, and you will have $p^2+6p=0.61+6.2q+q^2$; add 10, the coefficient of the third term, to both sides, and you will have $p^2+6p+10=10.61+6.2q+q^2$; multiply the former side by p, and the latter by $0.1+q$, and you will have $p^3+6p^2+10p=1.061+11.23q+6.3q^2+q^3$; lastly add -1, the numeral part, to both sides, and you will have $p^3+6p^2+10p-1=0.061+11.23q+6.3q^2+q^3$.

I chose to multiply the latter sides of the equations this way, that is, to begin with the numeral parts, not out of any necessity, but for convenience; because wherever I have a mind to stop, I can by this means more conveniently come at the simple powers of the root, without carrying on the multiplication to the production of useless terms: as for instance, suppose I had a mind that the foregoing operation should be the last, I should then have carried it on thus: $p=0.1+q$; therefore $p+6=6.1+q$; therefore $p^2+6p=0.61+6.2q$; therefore $p^2+6p+10=10.61+6.2q$; therefore $p^3+6p^2+10p=1.061+11.23q$; therefore $p^3+6p^2+10p-1=0.061+11.23q$.

This

This method of approximation is as applicable to all other equations as to cubics, provided a number can be found that approaches near enough to the true root; and this cannot be difficult to any one who knows how to make proper conjectures for that purpose; I mean, by substituting 0, 1, 10, 100, 1000, &c for the unknown quantity, and so finding out between what limits it consists; for when these limits are once known, it will be easy by a grosser sort of approximation to find out narrower limits, till we approach so near the root as to be able to form a regular process according to the directions given in art. 458.

This invention, (if it deserves the name of an invention,) I mean, of substituting 0, 1, 10, 100, 1000, &c for the unknown quantity in an equation in order to discover its limits, is commonly ascribed to one *Simon Stevin*, and is accordingly known by *Simon Stevin*'s rule for resolving all sorts of equations whatever; and whoever would know more of it, may see it further explained and exemplified in *Kersey*'s Algebra, book the second, chapter the tenth.

Of the resolution of biquadratic equations.

461. A biquadratic equation has been defined already to be an equation that rises to the fourth power of the unknown quantity; as if $x^4+px^3+qxx+rx+s=0$: where I must again advertise my reader, as I have oftener than once done before, that the signs $+$ do not so much signify that the terms of the equation are all affirmative, as that they are to be added all together, whether they be affirmative or negative.

Of this sort of equations the most simple form has been considered already, I mean, when the second and fourth terms are wanting; in which case, the resolution differs but little from that of a common quadratic: I shall therefore take notice of only one particular form more, whereof I shall give a particular solution, and then proceed to the resolution of all biquadratic equations whatever.

A LEMMA.

462. *Let there be two quadratic equations formed out of the powers of* a *and* x *(whereof* a *is supposed to be known and* x *to be unknown) in the manner following;* $xx+eax+aa=0$, *and* $xx+fax+aa=0$, *the extreme terms being* xx *and* aa *in both cases: I say then that if these two quadratic equations be multiplied together, they will produce a biquadratic equation of the following form,* $x^4+pax^3+qaaxx+pa^3x+a^4=0$, *where the coefficients of all the terms equidistant from the middle term on each side are the same.* Thus the coefficients of the first and last terms x^4 and a^4 are both equal to one, and those of the second and fourth terms pax^3 and pa^3x are both equal

equal to p: for if the equation $xx+eax+aa=0$ be multiplied into the equation $xx+fax+aa=0$, the equation thence arising will be $x^4+\overline{e+f}\times ax^3+\overline{2+ef}\times aaxx+\overline{e+f}\times a^3x+a^4=0$: make $e+f=p$, and $2+ef=q$, and the equation will then be $x^4+pax^3+qaaxx+pa^3x+a^4=0$.

463. *Hence* e converso, *whenever we have a biquadratic equation of this form,* $x^4+pax^3+qaaxx+pa^3x+a^4=0$, *where the coefficients of all the terms equidistant from the middle term on each side are the same, such an equation I say, with the limitation hereafter to be specified, may be resolved into two quadratics,* $xx+eax+aa=0$, *and* $xx+fax+aa=0$, *the four roots whereof will be the same with the four roots of the equation first proposed:* for we shall always have these two equations for determining the values of the two unknown coefficients e and f, to wit $e+f=p$, and $2+ef=q$; the former equation gives $f=p-e$, and the latter gives $f=\frac{q-2}{e}$; therefore $p-e=\frac{q-2}{e}$, and $pe-ee=q-2$, and $ee-pe=2-q$, and $ee-pe+\frac{1}{4}p^2=\frac{1}{4}p^2+2-q$: make $\frac{1}{4}p^2+2-q=ss$, and you will have $ee-pe+\frac{1}{4}p^2=ss$, and $e-\frac{1}{2}p=\pm s$, and $e=\frac{1}{2}p\pm s$, which is as much as to say, that if e be taken equal to $\frac{1}{2}p+s$, f will be equal to $\frac{1}{2}p-s$, because $e+f=p$. The rule then is this: *Let the equation proposed be* $x^4+pax^3+qaaxx+pa^3x+a^4=0$: *make* $\frac{1}{4}pp+2-q=ss$, $\frac{1}{2}p+s=e$, and $\frac{1}{2}p-s=f$; *and the equation proposed will resolve itself into these two quadratics, to wit,* $xx+eax+aa=0$, *and* $xx+fax+aa=0$.

Note first, If the coefficients of all the terms equidistant from the middle be not the same, the equation must not be looked upon as falling under the form above described: thus the equation $x^4+pax^3+qaaxx+pa^3x-a^4=0$ is not of the form above described, because the coefficients of the first and last terms are different, being $+1$ and -1; and the same may be observed of the coefficients of the second and fourth terms when they are either different, or have different signs before them.

2dly, If $a=1$, the equation will be changed into this, $x^4+px^3+qxx+px+1=0$; and therefore in every equation of this form, a may be supposed equal to 1.

3dly, If q be greater than $\frac{1}{4}pp+2$, the resolution will be impossible this way.

4thly, If the quantities e and f be found to have surds involved in them, they may however be taken as exactly as occasion requires.

A numeral example of the foregoing resolution.

Let the equation to be resolved be $x^4-8x^3+9x^2-8x+1=0$. Here $a=1$, $p=-8$, $q=+9$, $\frac{1}{4}p^2+2-q$ or $ss=9$, $s=3$, $\frac{1}{2}p+s$ or $e=-1$, $\frac{1}{2}p-s$ or $f=-7$;

$f=-7$; therefore the two quadratic equations into which the equation proposed resolves itself, are $xx-x+1=0$, and $xx-7x+1=0$: the two roots of the former equation are impossible, being $\frac{1+\sqrt{-3}}{2}$ and $\frac{\sqrt{-3}}{2}$; the roots of the latter equation are possible, being $\frac{7+\sqrt{45}}{2}$ and $\frac{7-\sqrt{45}}{2}$.

Sometimes it is not easy to discern at first sight, whether a biquadratic equation be of the foregoing form or not: as for instance, let the equation be $x^4+18x^3+98xx+162x+81=0$: now if this equation be of the foregoing form, a^4 will be equal to 81, and a to 3: let us then suppose $a=3$, and we shall have first $18x^3=6ax^3$, secondly $98xx=\frac{98}{9}aaxx$, thirdly $162x=6a^3x$, and the whole equation will now be $x^4+6ax^3+\frac{98}{9}aaxx+6a^3x+a^4$, which happens to be of the foregoing form; therefore $a=3$, $p=6$, $q=\frac{98}{9}$, $\frac{1}{4}p^2+2-q$ or $ss=\frac{1}{9}$, $s=\frac{1}{3}$, $\frac{p}{2}+s$ or $e=\frac{10}{3}$, $\frac{p}{2}-s$ or $f=\frac{8}{3}$, and the quadratic equations are $xx+\frac{10}{3}ax+a^2=0$, and $xx+\frac{8}{3}ax+aa=0$: but $\frac{10}{3}ax$ is $10x$, and $\frac{8}{3}ax$ is $8x$, and aa is 9; therefore the two quadratic equations into which the biquadratic equation proposed resolves itself are $xx+10x+9=0$, and $xx+8x+9=0$; therefore the four roots of the foregoing biquadratic equation are all possible, being -1, -9, $\overline{-4+\sqrt{7}}$ and $\overline{-4-\sqrt{7}}$.

A further illustration of the foregoing resolution by a geometrical problem out of Pappus, *book the seventh, propositions the 71st and 72d.* (Fig. 69.)

464. *Let* ABCD *be a given square whose opposite sides are* AD *and* BC: *It is required from the angle* A *to draw a line as* AE *to cut the side* DC *produced beyond* C *in* E, *and the side* BC *in* F, *so that the intercepted part* EF *may be equal to a line given.*

SOLUTION.

Call *AB* the side of the square a, *EF* the given intercepted part b; call also the unknown distance *DE* x, and consequently *AE* $\sqrt{xx+aa}$, and

and *CE* $x-a$, and the similar triangles *EDA* and *ECF* will give the following proportion, to wit, *ED* to *EA* as *EC* to *EF*, and consequently ED^2 to EA^2 as EC^2 to EF^2, that is, according to our notation, xx to $xx+aa$ as $xx-2ax+aa$ is to bb: multiply extremes and means, and you will have $bbxx=x^4-2ax^3+2aaxx-2a^3x+a^4$, and by transposition $x^4-2ax^3+\overline{2aa-bb}\times xx-2a^3x+a^4=0$: now to reduce this equation to the form of the last article, instead of making the middle term $\overline{2aa-bb}\times xx$, let us make it $\overline{2-\frac{bb}{aa}}\times aaxx$, and then the equation will stand in form thus; $x^4-2ax^3+\overline{2-\frac{bb}{aa}}\times aaxx-2a^3x+a^4=0$; where $p=-2$, $q=+2-\frac{bb}{aa}$, $\frac{1}{4}p^2+2-q$ or $ss=1+\frac{bb}{aa}=\frac{aa+bb}{aa}$: make $aa+bb=cc$, and you will have $ss=\frac{cc}{aa}$, and $s=\frac{c}{a}$, and $\frac{1}{2}p+s$ or $e=-1+\frac{c}{a}=\frac{-a+c}{a}$, and $\frac{1}{2}p-s$ or $f=\frac{-a-c}{a}$; therefore the two quadratic equations into which the first resolves itself will be these, $xx+\overline{\frac{-a+c}{a}}\times ax+aa=0$, and $xx\,\overline{\frac{-a-c}{a}}\times ax+aa=0$, or rather these, $xx\,\overline{-a+c}\times x+aa=0$, and $xx\,\overline{-a-c}\times x+aa=0$. Now as c is greater than a, (for it is equal to $\sqrt{aa+bb}$,) it is manifest that all the three terms of the former quadratic equation will be affirmative, that is, they will all have the same sign before them, and consequently in passing from the first term to the last there will be no change of signs; therefore both the roots of that equation will be negative; and for a contrary reason the two roots of the latter equation will be both affirmative; therefore we will consider this equation in the first place, to wit $xx\,\overline{-a-c}\times x+aa=0$. Make $a+c=2r$, and the equation will be $xx-2rx+aa=0$: now whoever considers this equation, will easily see, that without any further resolution it in a manner constructs itself; for if $xx-2rx+aa=0$, that is, if $2rx-xx=aa$, it is plain that a the side of the square must be a mean proportional between x and $2r-x$; therefore if the distance *DE* can be so taken in the line *DC* produced beyond *C*, that the side of the square shall be a mean proportional between *DE* and $2r-DE$, the point *E* will be rightly assigned, that is, *DE* will be one root of the equation, and the line *AE* will so cut the side *BC* in *F*, that the part *EF* shall be equal to b: but this

this may be easily effected in the following manner: produce AB beyond B to G, so that BG may be equal to c or $\sqrt{aa+bb}$, and consequently AG to $a+c$ or to $2r$; then if upon the diameter AG a semicircle as AEG be described, cutting the side DC produced beyond C in E, that intersection E will be the point required: for joining EG, and drawing EH perpendicular to AG, the triangle AEG will be right-angled at E, as being in a semicircle; therefore EH, which is equal to the side of the square, will be a mean proportional between AH and HG, that is, between DE and $2r-DE$ as was required. But the semicircle AEG will also cut the line DE in some other point as L, and the line DL will be under the same circumstances as the line DE, that is, the side of the square will be a mean proportional between DL and $2r-DL$, as may easily be seen by a like way of reasoning as before; therefore DL will be the other root of the equation, and a line as ALK drawn through L and cutting the side BC produced beyond C in K will have it's part $LK=b$. But for all this, as the problem was proposed, the point L will not solve it; for the point that solves the problem must lie in the line DC produced beyond C, whereas the point L lies between D and C, as I shall thus demonstrate.

In art. 108 it was demonstrated, that in a quadratic equation of this form and disposition, $Axx=Bx+C$, the product of the two roots multiplied together would always be equal to $-\frac{C}{A}$: now if this theorem be applied to our equation, to wit, $xx-2rx+aa=0$ or $xx=2rx-aa$, it will easily appear, that the rectangle of the two roots DE and DL will be equal to aa, that is, to DC^2; therefore of the two lines DE and DL, one must be greater, and the other less than DC; but DE is greater than DC *ex hypothesi*; therefore DL must be less, that is, the point L must lie between D and C, and consequently cannot solve the problem.

We come now to examine the other quadratic equation whereof the original biquadratic was composed, to wit, $xx\overline{-a+c}x+aa=0$. Make $c-a=2r$, and then the equation will be $xx+2rx+aa=0$, whose roots will both be negative as was before observed: but these negative roots may be easily changed into equal affirmative ones by changing the sign of x in the equation; and this may safely be done, provided these new roots be taken on a contrary side of D to the former ones: change then the sign of x in the equation $xx+2rx+aa=0$, and you will have $xx-2rx+aa=0$, an equation of the same form and construction with the former, making $2r=c-a$ instead of $c+a$ as before: produce then the line BA beyond A to g, and take Bg equal to c, and consequently $Ag=c-a$ or $2r$; and if a semicircle upon the diameter Ag cuts the line CD produced beyond

beyond D in two points e and l, the two lines De and Dl will be the other two roots of the original biquadratic equation; insomuch that if through the point A be drawn the lines eAf and lAk, cutting the side CB produced beyond B in f and k respectively, the intercepted lines ef and lk will each be equal to b: but if the point L would not solve the problem, much less will the points e and l do it.

That all these solutions might have been applicable to the foregoing problem, it ought to have been comprehended in more general terms thus: *Let* ABCD *be a given square whose opposite sides are* AD *and* BC: *It is required from the angle* A *to draw an indefinite line as* AE *cutting the sides* CD *and* CB *or those sides any way produced in* E *and* F *respectively, so that the intercepted part* EF *may be equal to* b. Had this problem, I say, been thus proposed, it would have admitted of four solutions instead of one: so much less confined is the nature of truth than that of human understanding; which by the narrowness of it's views often cramps even truth itself, as I have frequently observed in the course of this work.

N. B. In both these constructions both the semicircles AEG and Aeg are supposed to be situate on the same side of the line Gg with the square.

In this problem, that the two roots last found, to wit De and Dl, may be possible, the square of the given line EF or b must not be less than eight times the square BD, which I thus demonstrate: The equation $xx-2rx+aa=0$ gives $x=r\pm\sqrt{rr-aa}$; therefore r must not be less than a, nor consequently must $2r$ be less than $2a$; but $2r=c-a$ *ex hypothesi*; therefore $c-a$ must not be less than $2a$, nor c less than $3a$, nor cc less than $9aa$; but $cc=aa+bb$ *ex hypothesi*; therefore $aa+bb$ must not be less than $9aa$, nor bb less than $8aa$, that is, the square of the line EF or of ef must not be less than eight times the square BD. This is evident also from the foregoing construction: for if the diameter Ag be greater than $2AD$, the *radius* will be greater than AD, and so the semicircle reaching beyond the line De, will cut it in two points e and l; but the nearer the *radius* approaches in length to the line AD, the nearer will the two points of intersection e and l approach towards one another; if the *radius* be equal to the side AD, the line De will then become a tangent to the circle, and the two points of intersection e and l must now be looked upon as coalescing into one, and the roots De and Dl as equal: let now the *radius* be less than AD, and the semicircle will not reach the line CD produced, so that the points e and l, as well as the roots De and Dl will now become impossible; therefore that the roots De and Dl may be possible, the *radius* must not be less than AD, nor the diameter Ag less than $2AD$, that is $2r$ must not be less than $2a$; but it has been shewn before, that if $2r$ must not be less than $2a$, neither must the square of EF be less than eight times the square BD; therefore, that the roots

roots De and Dl may be poſſible, the ſquare of EF muſt be greater than eight times the ſquare BD. This then is another inſtance of the neceſſity of two roots becoming impoſſible together from a ſtate of equality. See art. 107, and art. 111 obſervation the third.

We may alſo further obſerve, that though there may be ſome particular caſes of a problem wherein ſome of the roots of the equation produced by it are impoſſible, yet there will always be other caſes of the ſame problem wherein they are all poſſible; and therefore the form of the equation ought ſtill to be retained, even when ſome of the roots are become of no uſe.

465. It very often happens that after we have any way arrived at a truth, we can then diſcover other avenues to the ſame truth more ſimple than thoſe we took, though perhaps not always more natural: this happens to be our caſe in the ſolution of the foregoing problem, where we repreſented the unknown diſtance DE by x, and ſo found that x had four ſignifications, to wit DE, DL, De, Dl; but in this ſolution it cannot but be obſerved, that the quantity BG has but two ſignifications, to wit $+$ or $-\sqrt{aa+bb}$: had we therefore made BG the unknown quantity in the ſolution of this problem, that is, had we made $BG=z$, and ſo formed a ſolution upon that poſition, we muſt neceſſarily at laſt have come to this very ſimple equation, $zz=aa+bb$, as will appear from the following ſolution.

Suppoſe the point E as determined, and let EG be perpendicular to AE, the reſt of the conſtruction continuing as before; make $BG=z$, and conſequently $AG=a+z$; make alſo $AF=y$, and conſequently $AE=y+b$, and the ſimilar triangles ABF and AEG will give AB to AF as AE to AG, that is, a to y as $y+b$ to $a+z$; whence by multiplying extremes and means we have $aa+az=yy+by$: further, as the triangle EHG is ſimilar to the triangle AEG, and that again ſimilar to the triangle ABF, it follows that the triangles EHG and ABF are ſimilar; and I ſay, equal too, becauſe their homologous ſides EH and AB are equal; therefore their other homologous ſides EG and AF are equal, that is, $EG=y$: ſince then the right-angled triangle AEG gives $AG^2=AE^2+EG^2$, we ſhall have $aa+2az+zz=yy+2by+bb+yy=2yy+2by+bb$: thus then we have two equations for determining the two unknown quantities z and y, to wit $aa+az=yy+by$, and $aa+2az+zz=2yy+2by+bb$; ſubtract twice the former equation from the latter, and you will have $zz-aa=bb$, and $zz=aa+bb$, and $z=\pm\sqrt{aa+bb}$; which is as much as to ſay, that if the point E be ſo taken in the line DCE, that the line EF ſhall have ſuch a length as the problem requires, and if moreover EG be perpendicular to AE, cutting AB produced in G, the part BG will be

be equal to $\sqrt{aa+bb}$; therefore *e converso*, if the point *G* be taken in the line *ABG*, so that *BG* shall be equal to $\sqrt{aa+bb}$, and if moreover the point *E* be so taken in the line *DCE*, that the angle *AEG* be a right angle, the line *EF* will have such a length as the problem requires; but the point *E* cannot exist in the line *DCE* and at the same time the angle *AEG* be right, unless *E* be one of the points where a circle upon the diameter *AG* cuts the line *DCE*; therefore if a circle upon the diameter *AG* cuts the line *DCE* in two points *E* and *L*, the lines *EF* and *LK* will have such a length as the problem requires. In like manner it may be demonstrated, that if the line *Bg* be taken equal to $-\sqrt{aa+bb}$, that is, if *Bg* be taken equal and contrary to *BG*, and if a circle upon the diameter *Ag* cuts the line *CD* produced in *e* and *l*, the lines *ef* and *lk* will also have such a length as the problem requires; so that the lines *EF*, *LK*, *ef* and *lk* will each be equal to *b*.

Of the resolution of all sorts of biquadratic equations by the mediation of cubics.

A LEMMA.

466. *Let there be two quadratic equations of the following form, to wit,* $xx+ex+f=0$, *and* $xx-ex+g=0$, *whose middle terms* $+ex$ *and* $-ex$ *are equal and contrary one to the other: I say then, that if these two equations be multiplied together, they will produce a biquadratic equation whose second term is wanting:* for they will produce the equation

$$x^4 \quad * \quad \begin{matrix}+f\\+g\\-ee\end{matrix}\, xx \quad \begin{matrix}+eg\\-ef\end{matrix}\, x \quad +fg=0.$$

467. *Hence* e converso, *whenever we have a biquadratic equation whose second term is wanting*, (and how to take away the second term of all sorts of equations I have shewn in art. 437,) *such an equation may by the help of the last article be resolved into two quadratic equations whose middle terms are equal and contrary one to the other.* Let the equation proposed be $x^4+qxx+rx+s=0$; I say that this equation may be resolved into these two quadratics, $xx+ex+f=0$, and $xx-ex+g=0$: for from the foregoing lemma we may draw three equations for determining the three unknown quantities *e*, *f* and *g* thus:

Equ. 1st, $f+g-ee=q$,
Equ. 2d, $eg-ef=r$,
Equ. 3d, $fg=s$.

From the first equation may be deduced

Equ. 4th, $g+f=q+ee$.

From

From the second, Equ. 5th, $g-f=\frac{r}{e}$.

Subtract the fifth equation from the fourth, and take half the remainder, and you will have

$$\text{Equ. 6th,}\quad f=\frac{q+ee-\frac{r}{e}}{2}.$$

Add the fourth and fifth equations together, and take half the the sum, and you will have

$$\text{Equ. 7th,}\quad g=\frac{q+ee+\frac{r}{e}}{2}.$$

Multiply the sixth and seventh equations together, and you will have $fg=\frac{q^2+2qee+e^4-\frac{rr}{ee}}{4}$: but $fg=s$ by the third equation; therefore we have $\frac{q^2+2qee+e^4-\frac{rr}{ee}}{4}=s$, and $q^2+2qee+e^4-\frac{rr}{ee}=4s$: multiply both sides by ee, and then put the equation into due form, and it will stand thus, $e^6+2qe^4\substack{+qq\\-4s}ee-rr=0$: put y for ee to reduce the bicubic to a cubic equation, and you will have $y^3+2qyy\substack{+qq\\-4s}y-rr=0$. The rule therefore is as follows: *Let the equation proposed be* $x^4+qxx+rx+s=0$, *and find* y *the root of this cubic equation* $y^3+2qyy\substack{+qq\\-4s}y-rr=0$; *then making* $\sqrt{y}=e$, $\frac{q+ee-\frac{r}{e}}{2}=f$, *and* $\frac{q+ee+\frac{r}{e}}{2}=g$, *the two quadratic equations into which the original one may be resolved will be* $xx+ex+f=0$, *and* $xx-ex+g=0$.

Of this method take the following example: let it be required to resolve the biquadratic equation $x^4-3xx-4x-3=0$ into two quadratics in order to find it's roots; where $q=-3$, $r=-4$ and $s=-3$: first then we must resolve this cubic equation $y^3-6yy+21y-16=0$; and to do this, we must take away the second term by substituting $z+2$ for y, and then the equation will be transformed into this, $z^3+9z+10=0$, whose root being extracted according to rules formerly given, we have $z=-1$, the other two roots being impossible: but if $z=-1$, then $z+2$ or $y=+1$, and $\sqrt{y}$ or $e=1$; whence $\frac{q+ee-\frac{r}{e}}{2}$ or $f=\frac{-3+1+4}{2}=+1$, and $\frac{q+ee+\frac{r}{e}}{2}$ or $g=\frac{-3+1-4}{2}=-3$; therefore the

the two quadratics sought are $xx+x+1=0$, and $xx-x-3=0$; the two roots of the former equation are impossible, being $\frac{-1\pm\sqrt{-3}}{2}$, and the roots of the latter are possible, being $\frac{+1\pm\sqrt{13}}{2}$.

468. For the better comprehending of what has been delivered in the last article, it may not be amiss to look a little into the composition of a biquadratic equation whose second term is wanting, as thus: let $x^4-27xx-14x+120=0$ be a biquadratic equation whose four roots are all possible, to wit $+2$, -3, -4 and $+5$, (for the sum of the affirmative roots must always be equal to the sum of the negative ones, to take away the second term, which in all these cases is supposed to be wanting:) now from what has heretofore been demonstrated concerning the nature of equations in general it is evident, that this of ours must be formed from a continual multiplication of these four simple equations, to wit $x-2=0$, $x+3=0$, $x+4=0$ and $x-5=0$: it is plain also, that out of these simple equations may be formed several combinations of quadratics two and two together, and that the two quadratics in every combination being multiplied together will produce the original equation: as *first*, if the roots $+2$ and -3, and the roots -4 and $+5$ be joined together, they will form these two quadratics, $\overline{x-2}\times\overline{x+3}=0$, and $\overline{x+4}\times\overline{x-5}=0$, that is, $xx+x-6=0$, and $xx-x-20=0$, where e the coefficient of the second term is ± 1: *secondly*, if the roots $+2$ and -4, and the roots -3 and $+5$ be joined together, there will be formed another combination of quadratics, as $\overline{x-2}\times\overline{x+4}=0$, and $\overline{x+3}\times\overline{x-5}=0$, that is, $xx+2x-8=0$, and $xx-2x-15=0$, where $e=\pm 2$: *thirdly*, if the roots 2 and 5 and the roots -3 and -4 be joined together, you will have $\overline{x-2}\times\overline{x-5}=0$, and $\overline{x+3}\times\overline{x+4}=0$, that is, $xx-7x+10=0$, and $xx+7x+12=0$, where $e=\pm 7$.

Besides these three combinations of quadratics, there can be formed no other; and therefore if every one of these combinations be represented indifferently by this general one, $xx+ex+f=0$ and $xx-ex+g=0$, the coefficient e can have no fewer than six different significations, to wit ± 1, ± 2, and ± 7; but it's square ee will have but three significations, to wit 1, 4 and 49; whence it follows, that whenever the equation $x^4-27xx-14x+120=0$ comes to be resolved into two quadratics by this method, the cubic equation by means whereof the quantity ee is determined, must have all it's roots possible, to wit 1, 4 and 49; and hence arises the necessity of the intervention of a cubic equation in the resolution of a biquadratic into two quadratics.

If all the poſſible roots of the cubic equation, by means whereof any biquadratic is to be ſplit, be negative, that is, if ee be negative, the quantities e and $\frac{r}{e}$ will be impoſſible, and ſo will all the quantities e, f and g; in which caſe all the four roots of the biquadratic propoſed will alſo be impoſſible: for the roots of the equation $x^2+ex+f=0$ are $\frac{-e\pm\sqrt{ee-4f}}{2}$, and the roots of the equation $xx-ex+g=0$ are $\frac{+e\pm\sqrt{ee-4g}}{2}$; therefore when the quantities e, f and g are impoſſible, theſe roots will be ſo too; therefore it can never be impoſſible to reſolve a biquadratic equation into two quadratics, whenever ſuch a reſolution can be of any uſe.

469. But there is one form of biquadratic equations which falls under the general one here reſolved, and which muſt by no means be overlooked in this place, becauſe it may perhaps puzzle an unexperienced Analyſt if ever he ſhould have the curioſity to attempt it this way; I mean a a biquadratic equation whoſe ſecond and fourth terms are both wanting: it may probably be objected, that the roots of ſuch an equation are beſt obtained by treating it as a quadratic, which is true; but I am not here ſo ſollicitous about finding the roots of ſuch an equation, as I am about gratifying the curioſity of my young Analyſt, by finding them according to the method laſt explained: for certainly if a rule be calculated for any general form of equations, it ought to be applicable to all particular forms that fall under that general one. Let then $x^4-5xx+4=0$ be an equation propoſed to be reſolved by this laſt method; where $q=-5$, $r=0$, and $s=+4$: here the cubic equation which muſt firſt be reſolved, is $y^3-10yy+9y-rr=0$, or becauſe $rr=0$, the equation will be $y^3-10yy+9y=0$: now as in this equation y is found in every ſignificant term, it is plain from the latter end of art. 429, that one of the values of y muſt be nothing: divide now the whole equation by y, in order to diſcover the other two roots, and you will have the equation $yy-10y+9=0$, whoſe roots are 9 and 1; therefore the three values of y, or ee in the above mentioned cubic equation were 9, 1 and 0; therefore the values of $+e$ in the reſolution will be 3, 1 and 0: if $e=3$, we ſhall have $\frac{q+ee-\frac{r}{e}}{2}$ or $f=+2$, and $\frac{q+ee+\frac{r}{e}}{2}$ or $g=+2$, becauſe r and conſequently $\frac{r}{e}=0$; and ſo the equations derived from this ſuppoſition will be $xx+3x+2=0$ and $xx-3x+2=0$: if $e=1$, we ſhall have $f=-2$, and $g=-2$, and the equations will be $xx+x-2=0$, and $xx-x-2=0$: if $e=0$, and conſe-

consequently $ee=0$, the quantities f and g will then be $\frac{q-\frac{r}{e}}{2}$ and $\frac{q+\frac{r}{e}}{2}$: but the chief difficulty in this case is to determine the value of the fraction $\frac{r}{e}$, whose terms are both equal to nothing; for though the quantities r and e, may be justly esteemed equally nothing in any case where they are considered in conjunction with finite quantities, yet when compared one with another, they may be in any ratio one to another, and the value of the fraction $\frac{r}{e}$ may be any thing, unless we can meet with some circumstance or other more than what we have hitherto taken notice of, to fix it. Now to forward our enquiry in this difficulty, let us first of all suppose the quantities r and e to be finite, but to pass in a finite time from something through all degrees of magnitude into nothing, so as to vanish both together: if then upon this supposition we can discover the ultimate ratio of r to e, we shall then at the same time see the ultimate magnitude of the fraction $\frac{r}{e}$, and so shall have all we want. And certainly there can be no absurdity or inconsistency in this supposition; for what absurdity can there be in supposing two quantities r and e to pass by degrees from something to nothing in a finite limited time? and if they do, there must be some such thing as a last instant of their existence, which will be the instant that terminates this limited time, though they cannot be supposed to have any last magnitude, because no limit is prescribed to their diminution: now if upon enquiry it shall be found, that in the very last instant of their existence, the quantity r shall be to the quantity e in a certain finite ratio, this is all we would be understood to mean by the ultimate ratio of r to e. To proceed then; the equation that expressed the relation of r to e or rather of r to y in general, whether finite or infinitely small, was this, $y^3-10yy+9y-rr=0$; where by y in the present case must be understood the least root of that equation: now by this equation

r

gethe

last instant of the existence of y, the first term of the equation which is y^3 will be infinitely less than the second which is $10yy$, because y which is nothing is infinitely less than 10 which is something: for the same reason the second term which is $10yy$ will be infinitely less than the third $9y$; therefore in our present case, the two last terms of the equation will give the last relation of r to y as effectually as the whole: let us then reject the two first terms of the equation as having no influence upon the conclusion,

conclusion, and we shall then have $9y - rr = 0$, and $y = \frac{rr}{9}$, and $\sqrt{y}$ or $e = \frac{r}{3}$: thus then at last we have found the ultimate ratio of r to e, which is that of r to $\frac{r}{3}$, that is, of 3 to 1, and the ultimate magnitude of the fraction $\frac{r}{e}$ is $\frac{3}{1}$ or 3 : this being known, we shall have $\frac{q - \frac{r}{e}}{2}$ or $f = -4$ and $\frac{q + \frac{r}{e}}{2}$, or $g = -1$, and the equations derived from this supposition of $e = 0$ will be $xx - 4 = 0$, and $xx - 1 = 0$.

How others may relish these mathematical mysteries I know not ; but for my own part I must own, it is not without the utmost pleasure and surprize that I observe, what admirable shifts nature hath to bring herself off upon all occasions when she seems to be surrounded with insuperable difficulties; shifts so far beyond all human contrivance, that it is not always in the power of human understanding to trace out her steps and follow her, much less to prescribe to her : but the sublimer parts of Mathematics, and more particularly the doctrine of Fluxions, furnish us with still more frequent and pregnant instances of this kind, insomuch that the ancient Sage (whether he knew it or not) had a great deal of reason on his side when he cried out as he did,

Magna est veritas, et praevalebit.

The Metaphysicians tell us, that the material world is nothing else but an emanation from the ideal: and it may be so, for ought I know ; but in the mean time I cannot but be surprized to hear Gentlemen talk at this high-flown rate, who know so little either of one world or the other. It is in Mathematics only that truth, that is, rational truth, appears most conspicuous, and shines in her strongest lustre: in all other sciences she is either self-evident, or lies so near the day as to afford but little pleasure in the discovery ; or if she lies deeper and must be dug for, she is seen for the most part through so much dross, obscurity and confusion, that falshood herself under a plausible disguise often passes for truth. Whosoever then would be thoroughly acquainted with the nature, beauty and harmony of truth; whosoever to the utmost of his finite capacity would see truth as it has actually existed in the mind of God from all eternity, he must study Mathematics more than Metaphysics : in Mathematics there appears an uninterrupted vein of truth, which the searcher is at liberty to pursue, or any particular branch of it, as far as he pleases, making every known truth subservient to the discovery of some other: but in most other

other sciences, all that beautiful analogy, all that harmonious connexion and consistency is quite lost; and those truths that are discovered, appear so scattered, and so very independent one of another, that they seem to have no manner of relation one to another, though it is certain that all truths have: upon all these accounts it is no wonder that much greater advances have been made in and by the Mathematical sciences, than in all the rest put together without them.

As to the material world, if nature be not governed by constant and steady laws, all Philosophy is vain and fruitless: but if on the contrary she be always consistent with herself, if she will sooner produce a monster than deviate in the least from any of her own laws, she will then submit in all cases to be examined and tried by those laws: whether they be such as fall immediately under common observation and experience, or others less obvious that are only by reason deducible from these, we shall always find nature as much obliged by her own laws, as truth by the necessity and reason of things; so that we never need to doubt of our following nature, if we reason justly from things known to things unknown, whether it be from effects to their causes, or from causes to their effects. But then we ought to be cautious in the several steps we take, and enquire first, whether the causes such and such *Phænomena* are to be ascribed to, have themselves a real existence in nature or not; and if they have, secondly, whether they are or can be naturally productive of those effects; and if they are, lastly, whether they be efficacious enough to produce them in such quantities and degrees as we find them in nature: by this means we shall find many effects that were vulgarly ascribed to different causes, actually proceeding from one and the same cause; and as our natural knowledge improves, the number of natural causes will be reduced, and nature, every step we take, will appear more simple and uniform. But then these deductions, these enquiries cannot possibly be made to any degree of certainty without a thorough knowledge of quantity and proportion: and hence arises the very great use and even necessity of the Mathematical sciences in natural Philosophy; and I doubt not but this was one main end of the allwise Author of nature in bestowing so inestimable a gift upon rational creatures; for this method of reasoning, if duely cultivated and pursued, would bring us nearer to a true knowledge of God in his creation than all the metaphysical jargon of the schools. This was the method the great *Newton* took; and how he succeeded in it is sufficiently known to all the sober and thinking part of the learned world.

It is not to be denied indeed but that Mathematicians may, and very often do fail in their researches after nature, sometimes from a too rash and inconsiderate application of their principles, but oftener from the difficulty of their subjects: but what then? because these Gentlemen sometimes fail in

in their attempts, must other minute Philosophers, that are wholly destitute of these main qualifications, think they have nature more in their power? Sure I am that these, as they have no foundation, so neither can they have the least colour of pretence to any such enquiries: all that these can do to keep their ignorance in countenance, must be seemingly to despise and heartily to rail at what they do not understand; as if they voluntarily declined a part of learning, which their want of application and genius have rendered them altogether unfit for.

But the undoubted experience of this and the last age have sufficiently established the use of Mathematical Learning in Mechanical and Natural Philosophy beyond all that has been or perhaps can be said against it. Nor have the Moderns, especially those of our own Nation, stopped here; for they have endeavoured to the utmost of their power to purge it from all those metaphysical disputes and subtilties wherewith it hath so long been overrun, poisoned and stifled in it's growth; subtilties which at best amount to little more than a sort of legerdemain tricks, contrived by Philosophers purely to cover their own ignorance by bewildering the minds of others, depraving their tastes, and often giving them such an unhappy turn of thought as utterly to disqualify them for all sober and rational enquiries: but as the evil we are here complaining of, and the malign influence it has upon the minds of those who are in any measure tainted with it, begins now (God be thanked) to be pretty well known, it is to be hoped that all wise men will guard against a distemper so easily catched, and so very difficultly cured. I do not know whether my zeal for truth may not have carried me too far in this digression: but I thought too much caution could not be given to all sober and ingenuous Youths, especially at their first setting out in their studies, against a false taste, which has done so much mischief in the world by extinguishing that exquisite and refined pleasure which mankind naturally feel in the contemplation of truth, where it is genuine and uncorrupted by a false alloy; a pleasure not founded in our appetites, passions or humours, but belonging to us purely as thinking, reasoning creatures; and if strictly enquired into, and it's abstracted nature thoroughly weighed, will perhaps be found the only one the present state of Man is capable of in common with superior Beings.

APPENDIX

APPENDIX.

MR. *Abraham de Moivre* having with great ſagacity not only diſcovered a method for extracting the cube root of an impoſſible binomial, ſuch as $a+\surd-b$, but alſo rendered that method univerſal, by ſhewing how to extract any other root of the ſaid binomial, and likewiſe how to extract any root out of any given power thereof, hath been pleaſed to communicate it, in order to it's being publiſhed by way of Appendix to theſe Elements. This important diſcovery is a freſh inſtance of the penetration and ſkill of the ingenious Author, and cannot but be very acceptable to thoſe who deſire to improve themſelves in Algebra. Dr. *Saunderſon* had formerly conſulted Mr. *de Moivre* upon the ſubject of extracting the cube root of an impoſſible binomial in the following Letter, which Letter was principally deſigned to return him thanks for the ſolution of the latter of his two curious problems about proportionals and for conſenting to have them both inſerted into his book; whereby alſo it appears, how highly our late Profeſſor eſteemed the great abilities of his learned Friend.

To Mr. *Abraham de Moivre.*

Dear Sir, *Cambridge, Septemb.* 26. 1738.

This is the firſt opportunity I could get to acknowledge the favour of your's, and to thank you for your ſolution of your ſecond problem, and for your having ſo frankly and without the leaſt reſerve, conſented to have them inſerted into my book, which will certainly be the greateſt ornament it can receive, and the greateſt recommendation of it to the world. In this laſt ſolution you have ſhewn (if poſſible) more art and penetration than in the former; but they are both maſter-pieces in their kind, and do a great deal of honour not only to yourſelf, but alſo to the art in general, by whoſe irreſiſtible (and I had almoſt ſaid, unlimited) power, ſuch great things can be effected, beyond the compaſs of all other ſciences, eſpecially when applied by one of your extraordinary ſagacity and genius. I find you and I have both hit upon the ſame thought in dividing one equation by another in order to obtain a more ſimple one; though you have made a much better uſe of it than I have done, or indeed had occaſion to do. Pray, when you write next, be ſo good as to let me know, whether you have any thing by you relating to the extraction

tion of the cube root of an impossible binomial, such as $-5+\sqrt{-2}$, or $-5-\sqrt{-2}$, or whether in your reading you have met with any way of doing this with the same certainty as in the case of a possible binomial: for my own part, I have met with nothing to the purpose about it, not even in *Wallis* himself, who attempts it. I am

Dear Sir,

most affectionately your's,

N. Saunderson.

P. S. I would not for all the world have been without this last solution, because they both together more compleatly illustrate your method, than would either of them alone; a method wherein you have so well succeeded, and which I am sure every one must make use of, that intends to penetrate any further into these difficulties.

To the Editor of Dr. *Saunderson*'s Algebra.

Sir,

Among several letters which I have received from my late worthy Friend Dr. *Saunderson*, there is one wherein he was pleased to put a question to me, which was, whether I had any method of extracting the cube root of an impossible binomial, such as $a+\sqrt{-b}$, adding, that what Dr. *Wallis* had writ upon that subject, did not satisfy him. I do not exactly remember the terms of my answer; but it was to this purpose, *viz.* that I had long ago demonstrated, that the rule given by Dr. *Wallis* was no more than a *petitio principii*, or at best, a trial which could not succeed in many cases; yet, that as my demonstration lay in a heap of many other papers, I could not readily come at it: but now that I have recovered it, I hereby send it you.

But it is necessary to premise one thing; which is, that the Doctor's design was to shew, that in cubic equations there is no case inexplicable, notwithstanding the common received opinion of the contrary. One of these cases appears in the equation by him mentioned, $r^3-63r=162$; wherein if we take the value of r according to the rule commonly ascribed to *Cardan*, we shall have $r=\sqrt[3]{81+\sqrt{-2700}}+\sqrt[3]{81-\sqrt{-2700}}$; which value consists of two parts, each of which includes the imaginary quantity $\sqrt{-2700}$. Now Dr. *Wallis* in order to prove the rashness of those who had asserted, that in cubic equations there are some cases irreducible, or (as he calls them) impracticable, says, that the cubic root of $81+\sqrt{-2700}$ may be extracted by another impossible binomial, *viz.* by

by $\frac{9}{2}+\frac{1}{2}\sqrt{-3}$; and in the same manner, that the cubic root of $81-\sqrt{-2700}$ may be extracted, and shews it to be $\frac{9}{2}-\frac{1}{2}\sqrt{-3}$; from whence he infers, that $\frac{9}{2}+\frac{1}{2}\sqrt{-3}$, $+\frac{9}{2}-\frac{1}{2}\sqrt{-3}$, or barely 9, is one of the roots of the equation proposed, *viz.* $r^3-63r=162$: he also finds the other two roots, -6, -3.

It must be owned, that $\frac{9}{2}+\frac{1}{2}\sqrt{-3}$ is the cubic root of $81+\sqrt{-2700}$, and that his inference is true: but those who will consult his Algebra, page 190 and 191, will plainly see, that the rule he gives is nothing but a trial, both in determining that part of the root which is out of the radical sign, and that part which is within; and that if the original equation had been such, as to have it's roots irrational, his trial would never have succeeded. But farther I shall prove, that the extracting the cube root of $81+\sqrt{-2700}$ is of the same degree of difficulty as that of extracting the root of the original equation $r^3-63r=162$, and that both require the trisection of an angle for a perfect solution: in order to which, let the following general problem be proposed.

To extract the cubic root of the binomial a+√—b.

SOLUTION.

Let the cube root of $a+\sqrt{-b}$ be $x+\sqrt{-y}$: therefore the cube of $x+\sqrt{-y}$ is equal to $a+\sqrt{-b}$; but the cube of it is $x^3+3xx\sqrt{-y}-3xy-y\sqrt{-y}$: make the rational parts equal to a, and the irrational equal to b: hence we have these two equations,

$$x^3-3xy=a,$$

$$\overline{3xx-y}\times\sqrt{-y}=\sqrt{-b};$$

and by squaring both equations, we shall have

$$x^6-6x^4y+9xxyy=aa,$$

and $$-9x^4y+6xxyy-y^3=-b:$$

then taking the difference of these two equations, we shall have

$$x^6+3x^4y+3xxyy+y^3=aa+b;$$

and extracting the cube root on both sides, we shall have $xx+y=\sqrt[3]{aa+b}$: let (for shortness's sake) $\sqrt[3]{aa+b}$ be supposed $=m$; we have therefore $y=m-xx$: but from the first equation, *viz.* $x^3-3xy=a$, we have $y=\frac{x^3-a}{3x}$; wherefore $m-xx=\frac{x^3-a}{3x}$, and $3mx-3x^3=x^3-a$, or $4x^3-3mx=a$: but it is known, that if r be the *radius*, l the cosine of an arc, and x the cosine of the third part of that arc, then the equation expressing the relation between l and x, will be $4x^3-3rrx=rrl$; which equation is of the same nature with the preceding, as appears by suppo-

sing

sing $rr=m$, or $r=\sqrt{m}$, and $rrl=a$, or $ml=a$, or $l=\frac{a}{m}$: and therefore we may now draw this conclusion, that if with a *radius* $=\sqrt{m}$, we describe a circle, and take an arc, of which the cosine shall be $\frac{a}{m}$, then shall the cosine of the third part of that arc be the value of x; which value being known, the value of y will also be known, it being always equal to $m-xx$, as we have shewn before.

Still we cannot be sure that the equation $4x^3-3mx=a$ depends upon the trisection of an arc, unless we know previously that the cube of m is greater than the square of a: but we may soon be satisfied of it, if we consider that m has been supposed $=\sqrt[3]{aa+b}$; therefore $m^3=aa+b$, which by inspection appears bigger than aa.

But before we proceed any farther, it will be proper to observe that there are three different values of x, and as many of y: for let C represent the whole circumference of the circle, of which the *radius* is $=\sqrt{m}$, and A the arc, of which the cosine is $\frac{a}{m}$; then the cosines of $\frac{A}{3}$, $\frac{C-A}{3}$, $\frac{C+A}{3}$ will be so many different values of x; and therefore there will be so many different values of y, it having been proved before that y is always $=m-xx$.

But to apply this to the case proposed by Dr. *Wallis*, which was to extract the cube root of $81+\sqrt{-2700}$; make $a=81$, and $b=2700$; therefore $aa+b=9261$, and consequently $m=\sqrt[3]{9261}=21$, and therefore our *radius* is $\sqrt{21}$. Again, the cosine $\frac{a}{m}=\frac{81}{21}=\frac{27}{7}$: now it will be found by an easy trigonometrical calculation, that if $\sqrt{21}$ be the *radius* of a circle, and $\frac{27}{7}$ the cosine of an arc, then this arc will be $32°.\ 42'$ nearly, representing the arc A; and therefore $C-A$ will be $327°.\ 18'$, and $C+A=392°.\ 42'$: then the thirds of those arcs will be $10°.\ 54'$, $109°.06'$, $130°.\ 54'$; whereof the first being less than a quadrant, it's cosine, that is, the sine of $79°.\ 06'$ must be looked upon as positive, and the other two being greater than quadrants, their cosines, that is, the sines of $19°.\ 06'$ and $40°.\ 54'$, must be looked upon as negatives. This being laid down, it will be found by the rules of trigonometry that the sines of $79°.\ 06'$, $19°.\ 06'$, $40°.\ 54'$, to *radius* $\sqrt{21}$, will respectively be 4.4999, -1.4999, -3.0000, or $\frac{9}{2}$, $-\frac{3}{2}$, -3; and consequently that the three values of y, which (as we have shewn

ſhewn before) are univerſally expreſſed by $m-xx$, will reſpectively be $21-\frac{81}{4}$, $21-\frac{9}{4}$, $21-9$, or $\frac{3}{4}$, $\frac{75}{4}$, 12; of which the ſquare roots are $\frac{1}{2}\sqrt{3}$, $\frac{5}{2}\sqrt{3}$, $2\sqrt{3}$; and therefore the three values of $\sqrt{-y}$ will be $\frac{1}{2}\sqrt{-3}$, $\frac{5}{2}\sqrt{-3}$, $2\sqrt{-3}$; from all which we may conclude, that the three roots of $\sqrt[3]{81+\sqrt{-2700}}$ are $\frac{9}{2}+\frac{1}{2}\sqrt{-3}$, $-\frac{3}{2}+\frac{5}{2}\sqrt{-3}$, $-3+2\sqrt{-3}$: and by the ſame way of arguing we may prove that the three roots of $\sqrt[3]{81-\sqrt{-2700}}$ are $\frac{9}{2}-\frac{1}{2}\sqrt{-3}$, $-\frac{3}{2}-\frac{5}{2}\sqrt{-3}$, $-3-2\sqrt{-3}$.

But to illuſtrate what I have ſaid, let it be propoſed to find the equation which would reſult from the addition of the two binomials $\sqrt[3]{a+\sqrt{-b}}+\sqrt[3]{a-\sqrt{-b}}$ in order to free them from their radicality: which to find, let us ſuppoſe

$$1°,\ z^3=a+\sqrt{-b};$$
$$2°,\ v^3=a-\sqrt{-b};$$
$$3°,\ z+v=x.$$

By the two firſt equations it appears that $z^3+v^3=2a$; by the third, that $z+v=x$; therefore $\frac{z^3+v^3}{z+v}=\frac{2a}{x}$; but $\frac{z^3+v^3}{z+v}=zz-zv+vv$, and therefore we may conclude that $zz-zv+vv=\frac{2a}{x}$; but ſquaring the third equation, we have $zz+2zv+vv=xx$; then ſubtracting the firſt of theſe two laſt equations from the ſecond, we ſhall have $3zv=xx-\frac{2a}{x}$; but if z^3 be multiplied by v^3, and $a+\sqrt{-b}$ by $a-\sqrt{-b}$, we ſhall have $v^3z^3=aa+b$; therefore $vz=\sqrt[3]{aa+b}$, and $3vz=3\sqrt[3]{aa+b}$; or ſuppoſing $\sqrt[3]{aa+b}=m$, then $3vz=3m$; but we had found before, that $3vz=xx-\frac{2a}{x}$; therefore $xx-\frac{2a}{x}=3m$, or $x^3-2a=3mx$, or $x^3-3mx=2a$; which to reſolve, the triſection of an angle is required, it being of the ſame nature as the former was, *viz.* $4x^3-3mx=a$; for if in the equation $x^3-3mx=2a$, inſtead of x we write $2x$, we ſhall have $8x^3-6mx=2a$, or $4x^3-3mx=a$.

But to go ſtill farther, let it be propoſed to extract ſuch root of the binomial $a+\sqrt{-b}$ as may be denominated by n, *viz.* $\sqrt[n]{a+\sqrt{-b}}$. Let $x+\sqrt{-y}$ be that root: then ſuppoſing $\sqrt[n]{aa+b}=m$, let a circle be deſcribed

scribed of which the *radius* shall be equal to $\sqrt{m}$: moreover supposing $\frac{n-1}{2}=p$, (no matter whether n be an odd or an even number, provided it be an integer,) take, in that circle, an arc, of which the cosine shall be $\frac{a}{m^p}$, and let that arc be supposed $=A$; let C represent the whole circumference; then the cosines of the arcs $\frac{A}{n}$, $\frac{C-A}{n}$, $\frac{C+A}{n}$, $\frac{2C-A}{n}$, $\frac{2C+A}{n}$ &c to the same *radius* $\sqrt{m}$, will be so many different values of x. But it is to be observed, that so many of those cosines are to be taken as there are units in n, and no more.

I have but one word to add, which is, that if it be required to extract any root out of any given power of the binomial $a+\sqrt{-b}$, for instance, the cube root of the square, the same may also be done thus: let $a+\sqrt{-b}$ be raised to it's square, and it will be $aa-b+2a\sqrt{-b}$; then supposing $aa-b=d$, and $2a\sqrt{-b}$ or $\sqrt{-4aab}=\sqrt{-e}$, nothing remains now to be done, but to extract the cube root of the binomial $d+\sqrt{-e}$, which may be done as before.

I think it would be needless to say any thing more upon this subject: wherefore I now subscribe myself

Sir, &c.

April 29. 1740. *A. De Moivre.*

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